Various Domination Parameters in Mycielski’s graphs

S. BALAMURUGAN¹, M. ANITHA² and N. ANBAZHAGAN³
¹PG Department of Mathematics, Government Arts College, Melur, Madurai, Tamilnadu, India
²Department of Mathematics, Syed Ammal Arts & Science College, Ramanathapuram, Tamilnadu, India
³Department of Mathematics, Alagappa University, Karaikudi, Tamilnadu, India

e-mail: balapoojaa2009@gmail.com, ani.thania@gmail.com, anbazhagan_n@yahoo.co.in

Abstract: Given a graph G and any integer m ≥ 0, Mycielski constructed a graph μ(G) and one can transform G into a generalized mycielskian of G, μm(G). This paper investigate the mycielskian number of μm(G) under the domination parameters strong domination, weak domination, dom chromatic, chromatic strong domination and chromatic weak domination. Also we show that for a graph G without isolated vertices, μ(G) and μm(G) are not strong(weak) efficient open dominatable whenever G is strong(weak) efficient open dominatable.

Key words: Mycielski’s Graph, Strong (Weak) Domination, Dom Chromatic set, Chromatic Strong(Weak) Domination, Strong(Weak) Efficient Open Domination.

1 Introduction

Let G = (V, E) be an undirected graph with vertex set V and edge set E. For graph theoretic terminology, we refer to [4] and [5]. The open neighborhood of v ∈ V is N(v) = {u ∈ V | uv ∈ E} and closed neighborhood of v ∈ V is N[v] = N(v) ∪ {v}. The strong open neighbourhood of a point u is the set Ns(u) consisting of all points v such that deg ≤ deg v which are adjacent with u. The strong neighbourhood is Ns[u] = Ns(u) ∪ {u}. The weak open neighbourhood of a point u is the set Nw(u) consisting of all points v such that deg ≥ deg v which are adjacent with u. The weak neighbourhood is Nw[u] = Nw(u) ∪ {u}.

E. Sampathkumar and L. Pushpalatha introduced the concepts of strong(weak) domination in [12]. A subset S of V(G) is called a strong dominating set of G if for every v ∈ V − S, there exists u ∈ S such that u and v are adjacent and deg u ≥ deg v. The strong domination numberγs(G) of G is the minimum cardinality of a strong dominating set. A subset S of V(G) is called a weak dominating set of G if for every v ∈ V − S, there exists u ∈ S such that u and v are adjacent and deg u ≤ deg v. The weak domination numberγw(G) of G is the minimum cardinality of a weak dominating set.

T. N. Janakiraman and M. Poobala ranjani [8] introduced a new conditional dom chromatic set and S. Balamurugan et al [2] extended this dom chromatic set to chromatic strong (weak) dominating set. A subset D of V is said to be a dom chromatic set if D is a dominating set and χ(< D >) = χ(G). The minimum cardinality of a dom chromatic set in a graph G is called the dom chromatic number and is denoted by γch(G). A dom chromatic set with cardinality γch is called γch − set of G. A subset D of V is said to be a...
chromatic strong dominating set if \( D \) is a strong dominating set and \( \chi(< D >) = \chi(G) \). The minimum cardinality of a chromatic strong dominating set in a graph \( G \) is called the chromatic strong domination number and is denoted by \( \gamma^c_s(G) \). A chromatic strong dominating set with cardinality \( \gamma^c_s \) is called \( \gamma^c_s - \) set of \( G \). A subset \( D \) of \( V \) is said to be a chromatic weak dominating set if \( D \) is a weak dominating set and \( \chi(< D >) = \chi(G) \). The minimum cardinality of a weak strong dominating set in a graph \( G \) is called the chromatic weak domination number and is denoted by \( \gamma^c_w(G) \). A chromatic weak dominating set with cardinality \( \gamma^c_w \) is called \( \gamma^c_w - \) set of \( G \).

We introduced the concept of strong(weak) efficient open domination in [1]. A subset \( D \) of \( V(G) \) is called a strong efficient open dominating set (or SEOD set, for short) of \( G \) if \( |N_v(D) \cap D| = 1 \), for every \( v \in V(G) \). A subset \( D \) of \( V(G) \) is called a weak efficient open dominating set (or WEOD set, for short) of \( G \) if \( |N_v(D) \cap D| = 1 \), for every \( v \in V(G) \). The strong (weak) efficient open domination number, denoted by \( \gamma_{ste}(G) \) (\( \gamma_{we}(G) \)), is the minimum cardinality of a strong (weak) efficient open dominating set of \( G \). We also call the corresponding set that \( \gamma_{ste} \) (\( \gamma_{we} \)) set of \( G \). A graph \( G \) is called a strong (weak) efficient open dominating graph or SEOD (WEOD) graph if it contains a strong(weak) efficient open dominating set. Also says that \( G \) is strong (weak) efficient open dominatable. In this paper we investigate the mycielskian number of \( \mu_m(G) \) under the domination parameters strong domination, weak domination, dom chromatic, chromatic strong domination and chromatic weak domination. Also we show that \( \mu(G) \) and \( \mu_m(G) \) are not strong(weak) efficient open dominatable whenever \( G \) is strong(weak) efficient open dominatable, for a graph \( G \) without isolated vertices.

2 The Mycielski Construction

In 1955, Mycielski, [7] introduced a admirable construction to increase the chromatic number of triangle free graphs without increasing a clique number. W. Lin et al [13] call this mycielski’s graph as mycielskian of \( G \).

The Mycielskian of a graph \( G \) is defined as follows:

Let \( G \) be a graph with vertex set \( V = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E \). Let \( V^1 \) be a copy of the vertex set and \( u \) be a single vertex. Then the Mycielskian \( \mu(G) \) has the vertex set \( V^0 \cup V^1 \cup \{u\} \). The edge set of \( \mu(G) \) is the set \( \{v^0_i, v^0_j : v_i v_j \in E\} \cup \{v^1_i : v_i \in V^1\} \cup \{v^1_i u : \forall v^1_i \in V^1\} \).

In general,

Let \( G \) be a graph with vertex set \( V = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E \) and let \( m \) be any positive integer. For each integer \( k(0 \leq k \leq m) \), let \( V^k \) be a copy of vertices in \( V \), that is \( V^k = \{v^k_1, v^k_2, \ldots, v^k_n\} \). The \( m \) - mycielskian \( \mu_m(G) \) has the vertex set \( V^0 \cup V^1 \cup \ldots \cup V^m \cup \{u\} \) where \( u \) is a single vertex. The edge set of \( \mu_m(G) \) is the set \( \{v^0_i, v^0_j : v_i v_j \in E\} \cup \left( \bigcup_{k=0}^{m-1} \{v^k_i v^{k+1}_j : v_i v_j \in E\} \right) \cup \{v^m_i u : \forall v^m_i \in V^m\} \).

W. Lin et al [13] define \( \mu_0(G) \) to be the graph obtained from \( G \) by adding a universal vertex \( u \).

We observe that every vertex \( v^k_i \) in \( V^k \) is adjacent to the vertices \( v^{k+1}_i \) in \( V^{k+1} \).
and $v_i^{k-1}$ in $V^{k-1}$, $k = 1, 2, ..., m - 1$ if $v_i$ is adjacent to $v_j$ in $G$. No two vertices in $V^k$ are adjacent to each other except $k = 0$ and $v_i^k$ and $v_j^l$ are not adjacent, for all $i, k, l$. Also, $\text{deg} v_i^j = 2 \text{deg} v_i$, for all $j = 0, 1, ..., m - 1$; $\text{deg} v_i^m = \text{deg} v_i + 1$ and $\text{deg} u = |V(G)|$.

We define the subset $A$ of $V(G)$ as $A = \{x| x^k \in A^k\}$ where $A^k$ is the subset of $V^k, (k = 0, 1, 2, ..., m - 1)$.

3 Various Dominations on Mycielski’s Graph

**Theorem 3.1**
For any graph $G$, $\gamma_s(\mu(G)) = \gamma_s(G) + 1$.

**Proof:**
Let $A$ be a $\gamma_s$-set of $G$. Then $A \cup \{u\}$ is a strong dominating set of $\mu(G)$. Hence $\gamma_s(\mu(G)) \leq \gamma_s(G) + 1$. Suppose $\gamma_s(\mu(G)) \leq \gamma_s(G)$. Let $D$ be a $\gamma_s$-set of $\mu(G)$. Then, $D$ must contain the vertex $u$. Otherwise no vertex in $D$ can dominate $u$. Now, let $D = A^0 \cup B^1 \cup \{u\}$. If $B^1$ is empty, then $D = A^0 \cup \{u\}$ implies that $A$ is a $\gamma_s$-set of $G$ with $|A| < \gamma_s(G)$. If $B^1$ is non empty, then $D_1 = A^0 \cup B^1 \cup \{u\}$ where $B^0 = \{x^0| x^1 \in B^1\}$. Since $B^0$ does not affect the condition of strong domination, $D_1$ is also $\gamma_s$-set of $\mu(G)$. Then $A \cup B$ is a $\gamma_s$-set of $G$ with the cardinality less than $\gamma_s(G)$. Since both cases leads to the contradiction, $\gamma_s(\mu(G)) = \gamma_s(G) + 1$, for any graph $G$. ■

**Theorem 3.2**
For any graph $G$, $\gamma_w(\mu(G)) + 1 \leq \gamma_w(\mu(G)) \leq 2 \gamma_w(G)$

**Proof:**
Let $D$ be a optimal weak dominating set of $\mu(G)$. Then $D = A^0 \cup B^1 \cup \{u\}$. If $B^1$ is empty, then all the vertices in $V^1$ may be weakly dominated by the vertices of $A^0$. Hence $\gamma_w(\mu(G)) = |D| = |A^0| + 1$. It clear that the set $A = \{x \in V(G)| x^0 \in V^0(G)\}$ is weakly dominating set of $G$. $|A| \geq \gamma_w(G)$. $\gamma_w(\mu(G)) \geq \gamma_w(G) + 1$. If $B^1$ is non empty, there exists a vertex $x^1 \in B^1$ such that $\text{deg} v^1 \leq \text{deg} u$. Hence $u \notin D$. $D = A^0 \cup B^1 \cup \{u\}$. $\gamma_w(\mu(G)) = |D| = |A^0 \cup B^1| \leq 2 \gamma_w(G)$. ■

**Theorem 3.3**
The Mycielskian number of a graph $G$ under the dom chromatic, strong chromatic and weak chromatic domination is $V(\mu(G))$.

**Proof:**
Let $D$ be the dominating set under the domination parameters such as dom chromatic, strong chromatic and weak chromatic dominating set of $\mu(G)$. Then, $\chi(D) = \chi(\mu(G))$. Since $\mu(G)$ is $\chi$-critical, the only dominating set under these domination parameter is the vertex set of $\mu(G)$.

$: \gamma_{ch}(\mu(G)) = V(\mu(G))$
$\gamma_s(\mu(G)) = V(\mu(G))$
$\gamma_w(\mu(G)) = V(\mu(G))$. ■
4 Various Dominations on Generalized Mycielskian Graph

Theorem 4.1
For a graph $G$, $\left\lceil \frac{m}{2} \right\rceil \gamma_s(G) + 1 \leq \gamma_s(\mu_m(G)) \leq m\gamma_s(G) + 1$

Proof:
Let $D$ be a $\gamma_s$ - set of $G$. Then $D^0$ may or may not be strongly dominate all the vertices of $V^1$.

Case : 1
Suppose if $D^0$ may not be strongly dominate all the vertices of $V^1$. Then $D^0 \cup D^1 \cup D^2 \cup ... \cup D^{m-1} \cup \{u\}$ is the strong dominating set of $\mu_m(G)$. $\gamma_s(\mu_m(G)) \leq [D^0 \cup D^1 \cup D^2 \cup ... \cup D^{m-1}] + 1 = m\gamma_s(G) + 1$

Case : 2
Suppose if $D^0$ may be dominate all the vertices in $V^1$. Then, $D^i \cup D^{i+1}$ is strongly dominate all the vertices in $V^{i-1} \cup V^i \cup V^{i+1} \cup V^{i+2}$. If $m = 4k, 4k - 1$, choose $2k$ appropriate strong dominating subsets $D^r$ and if $m = 4k + 1, 4k - 2$, choose $2k + 1$ appropriate strong dominating subsets $D^r$. Also, the vertex $u$ strongly dominates all the vertices in $V^m$. Hence $\gamma_s(\mu_m(G)) \geq \left\lceil \frac{m}{2} \right\rceil \gamma_s(G) + 1$.

Illustrative Example 4.2 For upper Bound,

Consider the graph $G = P_3$ then the strong dominating set $D = \{x\}$, where $x$ is a unique full degree vertex in $P_3$ and $\gamma_s(P_3) = 1$. It is clear that there exists no vertex $v$ to strongly dominate the vertex $x^i$ in each $V^i$, for $i = 0, 1, ..., m - 1$. Hence $S = \{x^0, x^1, ..., x^{m-1}, u\}$ is the $\gamma_s$ - set of $\mu_m(G)$. Also, $\gamma_s(\mu_m(G)) = m + 1 = m\gamma_s(G) + 1$

Illustrative Example 4.3 For lower Bound,

Consider the graph $G = P_4$ then the strong dominating set $D = \{x, y\}$, where $x$ and $y$ are supporting vertices in $P_4$ and $\gamma_s(P_4) = 2$. Let $S$ be a strong dominating set of $\mu_m(G)$. It is clear that the vertices $x^0, y^0$ strongly dominates all the vertices in $V^0 \cup V^1$ then for $m = 4k - 2$ choose $S = D^0 \cup D^3 \cup D^4 \cup D^5 \cup D^6 \cup D^7 \cup ... \cup D^{m-3} \cup D^{m-2} \cup \{u\}$; for $m = 4k - 1$ choose $S = D^0 \cup D^1 \cup D^4 \cup D^5 \cup D^6 \cup D^7 \cup ... \cup D^{m-3} \cup D^{m-2} \cup \{u\}$; for $m = 4k$ choose $S = D^1 \cup D^2 \cup D^5 \cup D^6 \cup D^7 \cup ... \cup D^{m-3} \cup D^{m-2} \cup \{u\}$ and for $m = 4k + 1$ choose $S = D^0 \cup D^2 \cup D^3 \cup D^6 \cup D^7 \cup ... \cup D^{m-3} \cup D^{m-2} \cup \{u\}$. Hence $\gamma_s(\mu_m(P_4)) = \left\lceil \frac{m}{2} \right\rceil \gamma_s(P_4) + 1$.

Illustrative Example 4.4 Consider the following graph $\mu_0(P_7)$
For this graph, $m = 8$; $\gamma_s(P_7) = 3$ and $\gamma_s(\mu_8(P_7)) = 18$. The encircled vertices form a $\gamma_s$ - set. Also, $\frac{m}{2} \gamma_s(G) + 1 = 13 \leq \gamma_s(\mu_8(P_7)) \leq 25 = m \gamma_s(G) + 1$

Theorem 4.5
For a graph $G$, $\gamma_w(\mu_m(G)) \leq (m + 1)\gamma_w(G)$ and
$$\gamma_w(\mu_m(G)) \geq \begin{cases} 
\left(\left\lfloor \frac{m}{2} \right\rfloor + 1 \right)\gamma_w(G) & \text{if miseven} \\
\left(\left\lceil \frac{m}{2} \right\rceil + 1 \right)\gamma_w(G) + 1 & \text{if misodd}
\end{cases}$$

Proof:
Let $D$ be a $\gamma_w$ - set of $G$. Then $D^0$ may or may not be weakly dominate all the vertices of $V^1$.

Case : 1
Suppose if $D^0$ may not be weakly dominate all the vertices of $V^1$. Then $D^0 \cup D^1 \cup D^2 \cup \ldots \cup D^{m-1} \cup D^m$ is the weak dominating set of $\mu_m(G)$. Therefore, $\gamma_w(\mu_m(G)) \leq |D^0 \cup D^1 \cup D^2 \cup \ldots \cup D^{m-1} \cup D^m| = (m + 1)\gamma_w(G)$

Case : 2
Suppose if $D^0$ may be dominate all the vertices in $V^1$. Then, $D^i \cup D^{i+1}$ is weakly dominate all the vertices in $V^{i-1} \cup V^i \cup V^{i+1} \cup V^{i+2}$. If $m = 4k, 4k + 1$, choose $2k + 1$ appropriate weakly dominating subsets $D^r$ and if $m = 4k - 1, 4k - 2$ , choose $2k$ appropriate weak dominating subsets $D^r$. Also, the vertex $v^m$ in $V^m$ weakly dominates the vertex $u$, for an odd $m, \gamma_w(\mu_m(G)) \geq \begin{cases} 
\left(\left\lceil \frac{m}{2} \right\rceil + 1 \right)\gamma_w(G) & \text{if miseven} \\
\left(\left\lfloor \frac{m}{2} \right\rfloor + 1 \right)\gamma_w(G) + 1 & \text{if misodd}
\end{cases}$. ■
**Illustrative Example 4.6** For upper Bound,

Consider the graph $G = P_4$, then the weak dominating set $D = \{x, y\}$, where $x$ and $y$ are pendant vertices in $P_4$ and $\gamma_w(P_4) = 2$. It is clear that there exists no vertex $v$ to weakly dominate the vertex $x^i, y^i$ of degree two in each $V^i$, for $i = 0, 1, ..., m - 1$. Hence $W = \{x^0, y^0, x^1, y^1, ..., x^m, y^m\}$ is the $\gamma_w$ - set of $\mu_m(G)$. Also, $\gamma_w(\mu_m(G)) = 2(m + 1) = (m + 1)\gamma_w(G)$

**Illustrative Example 4.7** For lower Bound,

Consider the graph $G = C_4$, then the weak dominating set $D = \{x, y\}$, where $x$ and $y$ are any two adjacent vertices in $C_4$ and $\gamma_w(C_4) = 2$. Let $W$ be a weak dominating set of $\mu_m(G)$. It is clear that the vertices $x^0, y^0$ weakly dominates all the vertices in $V^0 \cup V^1$, then for $m = 4k - 2$ choose $S = D^1 \cup D^2 \cup D^3 \cup D^4 \cup D^{m-1} \cup D^m$; for $m = 4k - 1$ choose $S = D^1 \cup D^2 \cup D^3 \cup D^4 \cup D^{m-1} \cup D^m \cup \{v^m\}$; for $m = 4k$ choose $S = D^0 \cup D^3 \cup D^4 \cup D^{m-1} \cup D^m$ and for $m = 4k + 1$ choose $S = D^0 \cup D^2 \cup D^3 \cup D^4 \cup D^5 \cup D^{m-2} \cup D^{m-1} \cup \{v^m\}$.

Hence $\gamma_w(\mu_m(G)) \geq \left\{ \begin{array}{ll} \left(\frac{m}{2} + 1\right)\gamma_w(G) & \text{if} \text{is even} \\ \left(\frac{m}{2} + 1\right)\gamma_w(G) + 1 & \text{if} \text{is odd} \end{array} \right.$

**Illustrative Example 4.8** Consider the following graph $\mu_8(P_8)$
For this graph, $m = 8$; $\gamma_w(P_8) = 3$ and $\gamma_w(\mu_8(P_8)) = 30$. The encircled vertices form a $\gamma_w$ - set. Also, $\left(\left\lceil \frac{m}{2} \right\rceil + 1 \right) \gamma_w(P_8) = 20 \leq \gamma_w(\mu_8(P_8)) \leq 36 = (m + 1) \gamma_w(P_8)$

**Theorem 4.9**

The generalized mycielskian graph, $\mu_m(G)$ of a strong efficient open dominating graph $G \neq K_2$ not a strong efficient open dominatable.

**Proof:**

Let $G$ be a strong efficient open dominating graph. Then $deg v \leq \Delta \leq \frac{n}{2}$. Let $\Delta_i$ denotes the maximum degree of the vertices $V^i \subseteq V(\mu_m(G))$, for $i = 0, 1, \ldots, m$. Then $\Delta_i \leq n$, for $i = 0, 1, \ldots, m - 1$; $\Delta_m \leq \frac{n}{2} + 1$ and $deg u = n$. Since $G \neq K_2$, $deg u \neq deg v^m_i$, for all $i$. Hence the strong neighborhood set of a vertex $u$ is empty. Hence $\mu_m(G)$ is not a strong efficient open dominatable. ■

**Corollary 4.10**

The mycielski’s graph $\mu(G)$ is not a strong efficient open dominatable, for a strong efficient open dominating graph $G$.

**Theorem 4.11**

The Mycielski’s graph $\mu_0(G)$ is not a strong efficient open dominatable.

**Theorem 4.12**

For a graph $G \neq K_2$, the generalized mycielskian graph, $\mu_m(G)$ is not a weak efficient open dominatable.

**Proof:**

Let $x$ be a vertex of $G$ such that $deg x = \delta(G)$. Then $deg x^m = \delta + 1$. Since $G \neq K_2$, $deg x^m \neq deg u$ and $deg x^m \neq deg y^m$, for all $y \in N(x)$, because $deg y^m \geq 2\delta$. Hence $N_w(x^m) = \phi$. Hence clearly, $\mu_m(G)$ is not a weak efficient open dominatable, for any graph $G$. ■

**Corollary 4.13**

For a graph $G \neq K_2$, the mycielski’s graph $\mu(G)$ is not a weak efficient open dominatable.

**Theorem 4.14**

The Mycielski’s graph, $\mu_0(G)$ is not a weak efficient open dominatable.

**Remark 4.15**

Let $G = K_2$, then $\mu_m(K_2) = C_{2m+3} \neq C_{4k}$. Hence $\mu_m(G)$ is neither efficiently open dominatable nor strong(weak) efficient open dominatable.

**References**


