Fuzzy KM-Ideals on K-Algebras

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Abstract

In the recent years, Fuzzy and Neuro-Fuzzy play an important role in many engineering and scientific research problems. Properties show significant implications on the real time application. In this paper, KM-ideals are introduced to investigate the properties of fuzzy KM-ideals and fuzzy setting of an ideal in K-algebra. Few of the properties are satisfied by our proposed KM-ideals. This paper also discusses application of K-algebras and Fuzzy KM-ideal in Cartesian product.

Keywords: K-algebras, KM-ideals, Fuzzy KM-ideals, Homomorphism, Cartesian product, Fuzzy relations.

1. Introduction

Algebra made on a group (G, e) with identity e and adjoined with binary operation ⊙ on G is presented by Dar and Akram [3]. With right identity e, algebra is non-commutative and non-associative. Abelian group is alike p-semisimple BCI on K-algebra and is proved in [1,3]. Authors convenience a K-algebra put up on a group as K(G) algebra[2]. The K(G) has been characterized by using right and left mappings in [2]. Recently, Akram and Dar [4] have further proved that the class of K(G)-algebras is a global class of B-algebras [5] when (G, e) is a nonabelian group, and they also proved that K(G)-algebra is a widespread class of the BCH/BCI/BCK-algebras[6,7,8] when (G, e) is an abelian group.

The fuzzy set theory is introduced and it is developed by Zadeh[9] and then others found the applications in many directions in various areas of sciences. The ideals of K-algebras and notions of sub algebras and fuzzy(maximal) are introduced by Akram et al. [10] and further studied by Jun et al. [11]. Some properties of the fuzzy ideals in a K-algebra are investigated[10,11,12,13]. In this paper KM-ideals and fuzzy KM-ideals are introduced and investigated to deal with homomorphism, Cartesian product of KM-ideals and strongest fuzzy relation.

2. Preliminaries

In this segment, we review various uncomplicated aspect that are obligatory for this paper. Let (G, e) be a group with the uniqueness e such that x2 ≠ e for some x(̸= e) ∈ G. A K-algebra built on G (briefly, K-algebra) is a structure K = (G, ⊙, e) where “⊙” is a binary operation on G which is induced from the operation “…”, that satisfies the following:

(k1) (∀a, x, y ∈ G) ((a ⊙ x) ⊙ (a ⊙ y) = (a ⊙ (y ⊙ x)) ⊙ a),

(k2) (∀a ∈ G)(a ⊙ (a ⊙ x) = (a ⊙ x) ⊙ a),

(k3) (∀a ∈ G)(a ⊙ e = a),

(k4) (∀a ∈ G)(e ⊙ a = a).

If G is abelian, then conditions (k1) and (k2) take the place of:

(k1') (∀a, x, y ∈ G) ((a ⊙ x) ⊙ (a ⊙ y) = y ⊙ x),

(k2') (∀a, x ∈ G)(a ⊙ (a ⊙ x) = x), correspondingly.
If the condition $\forall a, b \in H$ ($a \bigodot b \in H$) is satisfied then, a nonempty subset $H$ of a $K$-algebra $K$ is called a subalgebra of $K$. Note that every subalgebra of a $K$-algebra $K$ contains the identity $e$ of the group $(G, \cdot , e)$. A mapping $f : K_1 \to K_2$ of $K$-algebras is called a homomorphism if $f(x \bigodot y) = f(x) \bigodot f(y)$ for all $x, y \in K_1$. If $f$ is a homomorphism then $f(e) = e$. A nonempty subset $I$ of a $K$-algebra $K$ is called an ideal of $K$ if it satisfies the following condition:

(i) $e \in I$,
(ii) $\forall x, y \in G$ ($x \bigodot y \in I, y \bigodot (y \bigodot x) \in I \implies x \in I$).

Let $\mu$ be a fuzzy set on $G$, i.e., a map $\mu : G \to [0, 1]$. A fuzzy set $\mu$ in a $K$-algebra $K$ is called a fuzzy subalgebra of $K$ if it satisfies:

$$\forall x, y \in G \ (\mu(x \bigodot y) \geq \min\{\mu(x), \mu(y)\}).$$

Every fuzzy subalgebra $\mu$ of a $K$-algebra $K$ satisfies the following inequality:

$$\forall x \in G \ (\mu(e) \geq \mu(x)).$$

### 3. Fuzzy ideals of $K$-algebra

**Definition 3.1:**
A fuzzy set $\mu$ in a $K$-algebra $K$ is called a fuzzy ideal of $K$ if it satisfies:

(i) $\forall x \in G \ (\mu(e) \geq \mu(x))$,
(ii) $\forall x, y \in G \ (\mu(y) \geq \min\{\mu(y \bigodot x), \mu(x \bigodot (x \bigodot y))\}).$

**Theorem 3.2:**
Every fuzzy $KM$-ideal $\mu$ of $K$-algebra $X$ is order reversing; that is, for every $x, y \in X$. If $y \leq x$ then $\mu(y) \geq \mu(x)$.

**Proof:**
Let $\mu$ be a fuzzy $KM$-ideal of $K$-algebra $X$ and let $x, y \in X$ such that $y \bigodot x = 0$

$$\mu(y) \geq \min\{\mu(y \bigodot x), \mu(x \bigodot (x \bigodot y))\} \geq \min\{\mu(0), \mu(x)\} = \mu(x).$$

Hence $\mu(y) \geq \mu(x)$.

**Definition 3.3:**
Let $(X, \bigodot, 0)$ be a $K$-algebra, if the following conditions are satisfied. For all $x, y, z \in X$,

$$\mu(0) \geq \mu(x), \mu(y \bigodot z) \geq \min\{\mu(x \bigodot y), \mu(z \bigodot x)\}$$

then, a fuzzy subset $\mu$ in $x$ is called a fuzzy $KM$-ideal of $X$.

### 4. Homomorphism of $K$-algebra

In this section, we discussed about $KM$-ideals in $K$-algebra under homomorphism and its properties.

**Definition 4.1:**
Let $(X, 0, 0)$ and $(Y, \Delta, 0')$ be $K$-algebras. A mapping $f : X \to Y$ is called a homomorphism if $f(x \bigodot y) = f(x) \bigtriangleup f(y)$, for all $x, y \in X$.

**Definition 4.2:**
Let $f : X \to X$ be an endomorphism and $\mu$ be a fuzzy set in $X$. A new fuzzy set is defined by $\mu f$ in $X$ as $\mu f(x) = \mu f(x)$ for all $x$ in $X$.

**Definition 4.3:**
For any homomorphism $f : X \to Y$ the set $\{x \in X | f(x) = 0'\}$ is called the Kernel of $f$, denoted by $\text{Ker}(f)$ and the set $\{f(x) | x \in X\}$ is called the image of $f$ denoted by $\text{Im}(f)$. 

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**Theorem 4.4:**
Let $f$ be an endomorphism of a $K$-algebra $X$. If $\mu$ is a fuzzy KM-ideal of $X$, then so is $\mu_f$.

**Proof:**
Let $x, y \in X$,

$$\mu_f(x) = \mu(f(x))$$

$$\geq \min\{\mu(f(y \odot x)), \mu(f(y) \odot f(x \odot y))\}$$

$$= \min\{\mu(f(y \odot x)), \mu(f(x \odot (x \odot y)))\}$$

$$= \min\{\mu_f((y \odot x)), \mu_f(x \odot (x \odot y)))\}.$$ 

Hence $\mu_f$ is a fuzzy KM ideal of $X$.

**Theorem 4.5:**
Assume $(X, \odot, 0)$ and $(Y, \Delta, 0')$ be $K$-algebras. A mapping $f: X \rightarrow Y$ is a homomorphism of $K$-algebra. Then $\text{Ker}(f)$ is a KM-ideal.

**Proof:**
For any $(x, y) \in X \times X$, we have,

$$0 \leq f((y \odot x) \odot (x \odot (x \odot y))) = 0$$

and $f(x) = 0$.

$$f(y) \odot f(x) \Delta f(x) \Delta f(y)$$

$$= f(y) \odot (f(x) \Delta (0 \odot f(y)))$$

Therefore $f \in \text{Ker}(f)$.

Hence $\text{Ker}(f)$ is a KM-ideal.

**Theorem 4.6:**
For a given subset $\delta$ of a $K$-algebra $X$, let $R_\delta$ be the strongest fuzzy relation on $X$. If $\delta$ is a fuzzy KM-ideal of $X \times X$, then $R_\delta(X, X) \leq R_\delta(0, 0)$ for all $x \in X$.

**Proof:**
Given $R_\delta$ is the strongest fuzzy relation of $X \times X$, hence,

$$R_\delta(X, X) = \min\{\delta(x), \delta(x)\} \leq \min\{\delta(0), \delta(0)\} = R_\delta(0, 0),$$

which implies that $R_\delta(X, X) \leq R_\delta(0, 0)$.

**Theorem 4.7:**
For a given subset $\delta$ of a $K$-algebra $X$, let $R_\delta$ be the strongest fuzzy relation on $X$. If $R_\delta$ is a fuzzy KM-ideal of $X \times X$, then $\delta(X) \leq \delta(0)$ for all $x \in X$.

**Proof:**
Since $R_\delta$ is a fuzzy KM-ideal of $X \times X$, then $R_\delta(X, X) \leq R_\delta(0, 0)$ where $(0, 0)$ is the zero element of $X \times X$. This means that

$$\min\{\delta(x), \delta(y)\} \leq \min\{\delta(0), \delta(0)\},$$

which implies that $\delta(X) \leq \delta(0)$.

**Theorem 4.8:**
If $\mu$ and $\delta$ are fuzzy KM-ideals in a $K$-algebra $X$, then $\mu \times \delta$ is a fuzzy KM-ideal in $X \times X$.

**Proof:**
For any $(x, y) \in X \times X$, we have,

$$(\mu \times \delta)(0, 0) = \min\{\mu(0), \delta(0)\}$$
\[ \geq \min\{\mu(x), \delta(y)\} \]
\[ = (\mu \times \delta)(x, y) \]

Assume \((x_1, x_2), (y_1, y_2) \in X \times X\),
\[ (\mu \times \delta)(y_1, y_2) = \min\{\mu(y_1), \delta(y_2)\} \]
\[ \geq \min\{\min\{\mu(x_1 \circ x_1), \mu(x_1 \circ (x_1 \circ y_1))\}, \min\{\delta(y_2 \circ x_2), \delta(x_2 \circ (x_2 \circ y_2))\}\} \]
\[ \geq \min\{\min\{\mu(x_1 \circ x_1), \delta(y_2 \circ x_2)\}, \min\{\mu(x_1 \circ (x_1 \circ y_1)), \delta(x_2 \circ (x_2 \circ y_2))\}\} \]
\[ = \min\{\min\{(\mu \times \delta)((y_1 \circ x_1) / (y_1 \circ (x_1 \circ y_1)), (x_2 \circ (x_2 \circ y_2))\)\} \]

Therefore \(\mu \times \delta\) is a fuzzy KM-ideal in \(X\).

**Theorem 4.9:**
Let \(\mu\) and \(\delta\) be the fuzzy sets in a \(K\)-Algebra \(X\) such that \(\mu \times \delta\) is a fuzzy KM-ideal of \(X \times X\) then

(i) Either \(\mu(0) \geq \mu(x)\) or \(\delta(0) \geq \delta(x)\) for all \(x \in X\).

(ii) If \(\mu(0) \geq \mu(x)\) for all \(x \in X\) then either \(\delta(0) \geq \mu(x)\) or \(\delta(0) \geq \delta(x)\).

(iii) If \(\delta(0) \geq \delta(x)\) for all \(x \in X\) then either \(\mu(0) \geq \mu(x)\) or \(\mu(0) \geq \delta(x)\).

(iv) Either \(\mu\) or \(\delta\) is an fuzzy KM-ideal of \(X\).

**Proof:**
Assume \(\mu \times \delta\) is an fuzzy KM ideal in \(X \times X\), therefore \((\mu \times \delta)(0, 0) \geq (\mu \times \delta)(x, y)\) for all \((x, y) \in X \times X\).

\[ (\mu \times \delta)(y_1, y_2) \geq \min\{\min\{\mu(x_1 \circ x_1), \delta(y_2 \circ x_2)\}, \min\{\mu(x_1 \circ (x_1 \circ y_1)), \delta(x_2 \circ (x_2 \circ y_2))\}\} \]

for all \((x_1, x_2), (y_1, y_2) \in X \times X\).

(i) Suppose that \(\mu(0) < \mu(x)\) and \(\delta(0) < \delta(x)\) for some \(x, y \in X\).

\[ (\mu \times \delta)(x, y) = \min\{\mu(x), \delta(y)\} \]
\[ \geq \min\{\mu(0), \delta(0)\} \]
\[ = (\mu \times \delta)(0, 0). \]

This is an ambiguity. Therefore \(\mu(0) \geq \mu(x)\) or \(\delta(0) \geq \delta(x)\) for all \(x \in X\).

(ii) Imagine that there exist \(x, y \in X\) such that \(\delta(0) < \mu(x)\) and \(\delta(0) < \delta(x)\).

Then \((\mu \times \delta)(0, 0) = \min\{\mu(0), \delta(0)\} = \delta(0)\) and hence
\[ (\mu \times \delta)(x, y) = \min\{\mu(x), \delta(y)\} > \delta(0) \]
\[ = (\mu \times \delta)(0, 0) \]

which is a contradiction.

Hence, if \(\mu(0) \geq \mu(x)\) for all \(x \in X\) then either \(\delta(0) \geq \mu(x)\) or \(\delta(0) \geq \delta(x)\). Similarly we can prove that, if \(\delta(0) \geq \delta(x)\) for all \(x \in X\) then either \(\mu(0) \geq \mu(x)\) or \(\mu(0) \geq \delta(x)\) which yields (iii).

(iv) Initially, we prove that \(\delta\) is a fuzzy KM-ideal of \(X\). Since by (i) Either \(\mu(0) \geq \mu(x)\) or \(\delta(0) \geq \delta(x)\) for all \(x \in X\). Assume that \(\delta(0) \geq \delta(x)\) for all \(x \in X\).

Then \(\delta(x) = \min\{\mu(0), \delta(x)\} \]
\[ = (\mu \times \delta)(0, x) \]
\[ \delta(y) = \min\{\mu(0), \delta(y)\} \]
\[ = (\mu \times \delta)(0, y) \]
\[ \geq \min\{(\mu \times \delta)((0 \circ 0), (y \circ x)), (\mu \times \delta)((0 \circ 0), (x \circ (x \circ y)))\} \]
\[ \geq \min\{(\mu \times \delta)((0 \circ 0), (y \circ x)), (\mu \times \delta)(0, (x \circ (x \circ y)))\} \]
\[ = \min\{\delta(y \circ x), \delta(x \circ (x \circ y))\}. \]

Hence \(\delta\) is a fuzzy KM-ideal of \(X\).

Next we will establish that \(\mu\) is a fuzzy KM-ideal of \(X\).
Let \( \mu(0) \geq \mu(x) \). Since by (ii) \( \delta(0) \geq \mu(x) \) (or) \( \delta(0) \geq \delta(x) \). Assume that \( \delta(0) \geq \mu(x) \) then,
\[
\mu(x) = \min\{\mu(0), \delta(x)\} \\
= (\mu \times \delta)(x, 0) \\
\mu(y) = \min\{\mu(y), \delta(0)\} \\
= (\mu \times \delta)(y, 0) \\
\geq \min\{(\mu \times \delta)((y, 0) \circ (x, 0)), (\mu \times \delta)((x, 0) \circ (x \circ y)))\} \\
\geq \min\{(\mu \times \delta)((y \circ x), 0), (\mu \times \delta)(x \circ (x \circ y)))\} \\
= \min\{(\mu \times \delta)((y \circ x), 0), (\mu \times \delta)(x \circ (x \circ y)))\} \\
= \min\{\delta(y \circ x), \delta(x \circ (x \circ y)))\}.
\]
Hence \( \mu \) is a fuzzy KM-ideal of \( X \).

**Theorem 4.10:**
Assume \( \mu \) and \( \delta \) be a fuzzy subsets of a \( K \)-algebra \( X \) such that \( \mu \times \delta \) is a fuzzy KM-ideal of \( X \times X \). Then \( \mu \) or \( \delta \) is a fuzzy KM-ideal of \( X \).

**Proof:**
By Theorem 4.9(i), without loss of generality, we assume that \( \mu(x) \leq \mu(0) \) for all \( x \in X \).
From the Theorem 4.9(iii), it follows that, for all \( x \in X \), either \( \delta(0) \geq \mu(x) \) (or) \( \delta(0) \geq \delta(x) \). If \( \delta(0) \geq \mu(x) \) for all \( x \in X \), then \( \mu(0, x) = \min\{\delta(0), \mu(x)\} = \mu(x) \).
Let \( (x, y) \in X \times X \). Since \( \mu \times \delta \) is a fuzzy KM-ideal of \( X \) by Theorem 4.6, we get
\[
(\mu \times \delta)(0, 0) \geq (\mu \times \delta)(x, y) \leq (\mu \times \delta)(x \circ y).
\]
Using KM-ideal,
\[
(\mu \times \delta)(y \circ x) = \min\{\mu(y_1), \delta(y_2)\} \\
\geq \min\{\mu(y_1 \circ x_1), \mu(x_1 \circ (x_1 \circ y_1)), \delta(x_2 \circ (x_2 \circ y_2))\} \\
= \min\{\mu(y_1 \circ x_1), \delta((y_2 \circ x_2) \circ (x_2 \circ y_2))\} \\
= \min\{\mu(y_1 \circ x_1), \delta((y_2 \circ x_2) \circ (x_2 \circ y_2))\} \\
= \min\{\mu(y_1 \circ x_1), \delta((y_2 \circ x_2) \circ (x_2 \circ y_2))\}.
\]
In particular, if we take \( x_1 = y_1 = 0 \), then
\[
\delta(y_2) = (\mu \times \delta)(0, y_2) \geq \min\{\mu(0), \delta(0 \circ x_2), \delta(x_2 \circ (x_2 \circ y_2))\} \\
= \min\{\delta(0 \circ x_2), \delta(x_2 \circ (x_2 \circ y_2))\}.
\]
This verifies that \( \delta \) is a fuzzy KM-ideal of \( X \). The subsequent part is alike. Hence the theorem is proved.

5. **Conclusion**
In this paper, KM-ideals, fuzzy KM-ideals have been investigated to deal with homomorphism, Cartesian product of KM-ideal and strongest fuzzy relation. The properties of fuzzy KM-ideals and fuzzy setting of an ideal in K-algebra are verified using KM-ideals. We have checked few of the properties which are satisfied by our proposed KM-ideals. In addition, K-algebras are discussed and fuzzy KM-ideal is applied in Cartesian product.

**References**


