CHROMATIC WEAK DOMINATION ON CARTESIAN PRODUCT

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Abstract

A subset D of a vertex set V of a graph G is a chromatic weak dominating set if D is a weak dominating set and \(\chi(< D >) = \chi(G)\). The minimum cardinality of a chromatic weak dominating set is called the chromatic weak domination number of G and is denoted by \(\gamma_w(G)\). In this paper we calculating the chromatic weak domination number for product graphs using Cartesian product.

Keywords: Domination, Weak domination, Chromatic weak domination, Chromatic Weak domination number.

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1. Introduction

Let \(G = (V,E)\) be finite simple, undirected graph. A dominating set D is said to be a strong dominating set if \(\text{deg}(u) \geq \text{deg}(v)\). The minimum cardinality of a strong dominating set is called the strong domination number of G and is denoted by \(\gamma_s(G)\). A coloring of a graph G is an assignment of colors to all its vertices. A proper coloring is a coloring in which adjacent vertices are assigned different colors. The chromatic number \(\chi(G)\) is the minimum number of colors necessary to give a proper coloring of G. A set D contains in V is said to be a dom-chromatic set if D is a dominating set and \(\chi(< D >) = \chi(G)\). The minimum cardinality of a dom-chromatic set is called the dom-chromatic number of G and is denoted by \(\gamma_{ch}(G)\). A set D contains in V is said to be a chromatic weak dominating set if D is a weak dominating set and \(\chi(< D >) = \chi(G)\). The minimum cardinality of a chromatic weak dominating set is called the chromatic weak domination number of G and is denoted by \(\gamma_w(G)\). In this paper we are going to find the chromatic weak domination number for some cartesian product graphs. Here we are combining the graphs such as path, cycle, star with each other and find the chromatic domination number for that graphs. The concept is first
introduced by Wilfried Imrich and Sandi Klavzar. It was developed by T.N Janakiraman and M.Poobalaranjani. She extended the concept of cartesian product in graphs under dom-chromatic concept. Here we calculated the chromatic weak domination number for product graphs.

**Definition 1.1** The Cartesian product $G \Box H$ of two graphs $G$ and $H$ is defined on the cartesian product $V(G) \times V(H)$ of the vertex sets of the factors. The edge set $E(G \Box H)$ is the set of pairs $((u, v), (x, y))$ of vertices for which $u = x$ and $(v, y) \in E(H)$ or $(u, x) \in E(G)$ and $v = y$. Thus $V(G \Box H) = V(G) \times V(H)$, $E(G \Box H) = \{((u, v), (x, y)) | u = x, (v, y) \in E(H) \text{ or } (u, x) \in E(G), v = y\}$.

**Theorem 1.2** $\gamma_w^c(K_{1,m} \Box K_{1,n}) = mn + 1; m, n \geq 2$.

*Proof.* Let $\{v_1, v_2, \ldots, v_{m+1}\}$ be the vertex set of $K_{1,m}$ with the full degree vertex $v_1$ and $\{u_1, u_2, \ldots, u_{n+1}\}$ be the vertex set of $K_{1,n}$ with the full degree vertex $u_1$. Then, the vertex set of $K_{1,m} \Box K_{1,n}$ is $V(K_{1,m} \Box K_{1,n}) = \{(v_i, u_j) / 1 \leq i \leq m + 1, 1 \leq j \leq n + 1\}$ clearly $K_{1,m}$ and $K_{1,n}$ has the non increasing order of degree sequence $(m, 1,1,\ldots,1)$ and $(n, 1,1,\ldots,1)$ respectively. Then the degree sequence of $(K_{1,m} \Box K_{1,n})$ is

$$(m + n, m + 1, m + 1, \ldots, m + 1, n + 1, n + 1, \ldots, n + 1, 2, 2, \ldots, 2)$$

Let $D_1 = \{(v_i, u_j) / 2 \leq i \leq m + 1, 2 \leq j \leq n + 1\}$ Then, clearly $D_1$ weakly dominate all other vertices of $V(K_{1,m} \Box K_{1,n})$ except $(v, u_1)$. For chromaticity, we have to choose a unique vertex say $(x, y)$ from the set $\{(u_1, v_1) / 2 \leq i \leq m + 1\} \cup \{(u_i, v_1) / 2 \leq i \leq n + 1\}$ which is adjacent to a vertex in $D_1$, since $\chi(K_{1,m} \Box K_{1,n}) = 2$. Let $D = D_1 \cup \{(x, y)\}$ then $D$ is the chromatic weak dominating set of $(K_{1,m} \Box K_{1,n})$. Therefore $\gamma_w^c(K_{1,m} \Box K_{1,n}) \leq |D_1| + 1 = mn + 1$. Therefore $\gamma_w^c(K_{1,m} \Box K_{1,n}) = mn + 1$.

Suppose $\gamma_w^c(K_{1,m} \Box K_{1,n}) < |D|$, clearly $\chi(< D >) = 2$. If we remove a vertex $v$ from $D_1$ then there exist no vertex $u$ such that $v$ can weakly dominated by $v$. Also, suppose we remove the vertex $(x, y)$ then $\chi(< D >) = 1 < 2 = \chi(G)$. Hence $\gamma_w^c(K_{1,m} \Box K_{1,n}) = |D|$. Therefore $\gamma_w^c(K_{1,m} \Box K_{1,n}) = mn + 1$.

**Theorem 1.3** $\gamma_w^c(K_{1,3} \Box P_n) = n + 4$.

*Proof.* Let $P_n$ be a path with vertex set $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and let $\{u_1, u_2, u_3, u_4\}$ be the vertex set of $K_{1,3}$ with the full degree vertex $u_1$. Then $V(K_{1,3} \Box P_n) = \{(u_i, v_j) / 1 \leq i \leq 4, 1 \leq j \leq u\}$ Let $D_1 = \{(u_2, v_1), (u_3, v_1), (u_4, v_1), (u_2, v_2), (u_3, v_2), (u_4, v_3)\}$ Let $D_2$ be set of all vertices chosen as follows. For each $j$, choose a vertex $(u_i, v_j)$ such that $d(v_j, v_{j+1}) \leq 3$, for all $j_1, j_2 \in 1, 2, \ldots, n$ and for $i = 1, 2, 3$. Clearly $D_1 \cup D_2$ is a chromatic weak dominating set of $K_{1,3} \Box P_n$. Therefore $\gamma_w^c(K_{1,3} \Box P_n) \leq |D_1 \cup D_2| = |D_1| + |D_2| = 6 + n - 2 = n + 4$. Suppose $\gamma_w^c(K_{1,3} \Box P_n) < |D_1 \cup D_2|$. Clearly $\chi(< D_1 \cup D_2 >) = 2$. If we remove a vertex $v$ from $D_1$, then there exist no vertex $u$ such that $v$ can weakly dominated by $v$. Also, suppose we remove the vertex $(u_i, v_j)$ either $i = 1$ or $i = 2$ or $i = 3$ and $j = 2$ to $n - 1$ then
\[(u_2, v_j) \in D_2, \text{ since } (u_2, v_j) \text{ cannot be dominated by the vertex of } D_1 \text{ and } D_2. \text{ Since both cases leads to the contradiction, Hence } \gamma_w^c(K_{1,3} \Box P_n) = |D_1 \cup D_2|. \text{ Therefore } \gamma_w^c(K_{1,3} \Box P_n) = n + 4.\]

**Theorem 1.4** \[\gamma_w^c(P_3 \Box P_n) = n + 2.\]

**Proof.** Let \(P_n\) be a path with vertex set \(V(P_n) = \{v_1, v_2, \ldots, v_n\}\) and \(V(P_3) = \{u_1, u_2, u_3\} \text{ Then } V(P_3 \Box P_n) = \{(u_i, v_j) / 1 \leq i \leq 3, 1 \leq j \leq n\} \text{ Let } D_1 = \{(u_1, v_j), (u_1, v_n), (u_3, v_j), (u_3, v_n)\} \text{ and } D_2 = \{(u_1, v_1), (u_2, v_j), (u_2, v_n), (u_3, v_1), (u_3, v_2) \ldots (u_n, v_n)\} \text{ either } i = 1 \text{ or } i = 3 \text{ and } j = 2 \text{ to } n-1 \text{ then } (u_2, v_j) \in D_2, \text{ since } (u_2, v_j) \text{ cannot be dominated by the vertex of } D_1 \text{ and } D_2. \text{ Since both cases leads to the contradiction. Hence } \gamma_w^c(P_3 \Box P_n) = |D_1 \cup D_2|. \text{ Therefore } \gamma_w^c(P_3 \Box P_n) = n + 2.\]

**Remark 1.5** Let us consider the following grid subgraph \(H\) of \(P_n \Box P_m\) with the vertex set \((x_i, y_j)/1 \leq i \leq n, 1 \leq j \leq m.\)

Let \[S_1(i) = (x_{i+1}, y_{j+1}), (x_{i+1}, y_{j+2}) \text{ or } (x_{i+2}, y_{j+1}), (x_{i+2}, y_{j+2}) \text{ or } (x_{i+2}, y_{j+2}), (x_{i+4}, y_{j+2}) \text{ and } S_2(i) = (x_{i+1}, y_{j+1}), (x_{i+2}, y_{j+2}) \text{ or } (x_{i+2}, y_{j+2}), (x_{i+4}, y_{j+2})\]

Then clearly \(S_1(i)\) is a Chromatic weak dominating set of \(H\) and \(S_2(i)\) is a weak dominating set of \(H.\)

**Theorem 1.6** For \(n \geq 4\) and for \(k \in N\gamma_w^c(P_3 \Box P_n) = \begin{cases} 4k + 2 & \text{if } n = 3k + 1 \\ 4k + 4 & \text{if } n = 3k + 2 \\ 4k + 5 & \text{if } n = 3k + 3 \end{cases}\]

**Proof.** Let \(P_n\) be a path with vertex set \(V(P_n) = \{v_1, v_2, \ldots, v_n\}\) and \(V(P_3) = \{u_1, u_2, u_3, u_4\}\) Then \(V(P_3 \Box P_n) = \{(u_i, v_j) / 1 \leq i \leq 4, 1 \leq j \leq n\}.\)

**Case 1:** \(n = 3k+1\)

Let \(D_1 = \{(u_1, v_1), (u_1, v_n), (u_3, v_1), (u_3, v_n)\}\),

\(D_2 = \{(u_1, v_1), (u_2, v_4), (u_2, v_7), (u_4, v_7), \ldots, (u_1, v_{3i+1}), (u_4, v_{3i+1}), \ldots, (u_1, v_n), (u_4, v_n)\}\),

\(D_3 = U S(i) \text{ where } S(i) = S^{(i)}(1) \text{ or } S^{(i)}(2) \text{ and } S^{(i)} = S^{(i)}(1) \text{ for atleast one } i.\)

Let \(D = D_1 U D_2 U D_3.\) Then, clearly \(D\) is a chromatic weak dominating set of \(P_3 \Box P_n\) Therefore \(\gamma_w^c(P_3 \Box P_n) \leq |D| = 4 + (2k - 1) + 2k = 4 + 2k.\)

**Case 2:** \(n = 3k+2\)

Let \(D_1 = \{(u_1, v_1), (u_1, v_n), (u_4, v_1), (u_4, v_n)\}.\) For each \(i = 1, 4, \)
\[D_2 = \{(u_i, v_4), (u_i, v_7), \ldots, (u_i, v_{3t+1}), (u_i, v_{3t+2}), \ldots, (u_i, v_{n-3})\} \text{ for } t = 1, 2, \ldots, k,\]

\[D_3 = \bigcup S^{(i)} \text{ where } S^{(i)} = S_1^{(i)} \text{ or } S_2^{(i)} \text{ and } S^{(i)} = S_3^{(i)} \text{ for at least one } i.\]

Let \(D = D_1 \cup D_2 \cup D_3.\) Then, clearly \(D\) is a chromatic weak dominating set of \(P_4 \square P_n.\)

Therefore \(\gamma_w^c(P_4 \square P_n) \leq |D| = 4 + 2k + 2k = 4 + 2k + 2k = 4k + 4.\)

**Case 3:** Let \(n = 3k+3\)

Let \(D_1 = \{(u_1, v_1), (u - 1, v_n), (u_4, v_1), (u_4, v_n)\}.\) For each \(i = 1, 4, D_2 = \{(u_i, v_4), (u_i, v_7), \ldots, (u_i, v_{3t+1}), (u_i, v_{3t+2}), \ldots, (u_i, v_{n-6}), (u_i, v_{n-3})\} \text{ for } t = 1, 2, \ldots, kD_3 = \bigcup S^{(i)} \text{ where } S^{(i)} = S_1^{(i)} \text{ or } S_2^{(i)} \text{ and } S^{(i)} = S_3^{(i)} \text{ for at least one } i. \quad D_4 = \{(u_2, v_{3t+2})\} \text{ or } \{(u_3, v_{3t+2})\} \text{ for } t = 1, 2, \ldots, k \quad \text{Let } D = D_1 \cup D_2 \cup D_3 \cup D_4.\) Then, clearly \(D\) is a chromatic weak dominating set of \(P_4 \square P_n.\) Therefore \(\gamma_w^c(P_4 \square P_n) \leq |D| = 4 + 2k + 2k + 1 = 4k + 5.\) Conversely, Suppose \(\gamma_w^c(P_4 \square P_n) < |D|\) clearly \(\chi(<D>) = 2.\) If we remove a vertex \(v\) from \(D\) then there exist no vertex \(u\) such that \(v\) can weakly dominated by \(v.\) Also, suppose we remove the vertex \((x, y),\) then \(\chi(<D>) = 1 < 2 = \chi(<G>).\)

Hence \(\gamma_w^c(P_4 \square P_n) = |D|. \quad \gamma_w^c(P_4 \square P_n) = \begin{cases} 4k + 2 & \text{if } n = 3k + 1 \\ 4k + 4 & \text{if } n = 3k + 2 \\ 4k + 5 & \text{if } n = 3k + 3 \end{cases}\)

**References**


