

CHROMATIC WEAK DOMINATION ON CARTESIAN PRODUCT

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Abstract

A subset D of a vertex set V of a graph G is a chromatic weak dominating set if D is a weak dominating set and $\chi(< D >) = \chi(G)$. The minimum cardinality of a chromatic weak dominating set is called the chromatic weak domination number of G and is denoted by $\gamma_w^c(G)$. In this paper we calculate the chromatic weak domination number for product graphs using Cartesian product.

Keywords: Domination, Weak domination, Chromatic weak domination, Chromatic Weak domination number.

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1. Introduction

Let $G = (V, E)$ be finite simple, undirected graph. A dominating set D is said to be a strong dominating set if $\deg(u) \geq \deg(v)$. The minimum cardinality of a strong dominating set is called the strong domination number of G and is denoted by $\gamma_s(G)$. A coloring of a graph G is an assignment of colors to all its vertices. A proper coloring is a coloring in which adjacent vertices are assigned different colors. The chromatic number $\chi(G)$ is the minimum number of colors necessary to give a proper coloring of G . A set D contains in V is said to be a dom - chromatic set if D is a dominating set and $\chi(< D >) = \chi(G)$. The minimum cardinality of a dom - chromatic set is called the dom - chromatic number of G and is denoted by $\gamma_{ch}(G)$. A set D contains in V is said to be a chromatic weak dominating set if D is a weak dominating set and $\chi(< D >) = \chi(G)$. The minimum cardinality of a chromatic weak dominating set is called the chromatic weak domination number of G and is denoted by $\gamma_w^c(G)$. In this paper we are going to find the chromatic weak domination number for some cartesian product graphs. Here we are combining the graphs such as path, cycle, star with each other and find the chromatic domination number for that graphs. The concept is first

introduced by Wilfried Imrich and Sandi Klavzar. It was developed by T.N Janakiraman and M. Poobalaranjani. She extended the concept of cartesian product in graphs under dom-chromatic concept. Here we calculated the chromatic weak domination number for product graphs.

Definition 1.1 The Cartesian product $G \square H$ of two graphs G and H is defined on the cartesian product $V(G) \times V(H)$ of the vertex sets of the factors. The edge set $E(G \square H)$ is the set of pairs $((u, v), (x, y))$ of vertices for which $u = x$ and $(v, y) \in E(H)$ or $(u, x) \in E(G)$ and $v = y$. Thus $V(G \square H) = V(G) \times V(H)$, $E(G \square H) = \{(u, v), (x, y) \mid u = x, (v, y) \in E(H) \text{ or } (u, x) \in E(G), v = y\}$.

Theorem 1.2 $\gamma_w^c(K_{1,m} \square K_{1,n}) = mn + 1; m, n \geq 2$.

Proof. Let $\{v_1, v_2, \dots, v_{m+1}\}$ be the vertex set of $K_{1,m}$ with the full degree vertex v_1 and $\{u_1, u_2, \dots, u_{n+1}\}$ be the vertex set of $K_{1,n}$ with the full degree vertex u_1 .

Then, the vertex set of $K_{1,m} \square K_{1,n}$ is $V(K_{1,m} \square K_{1,n}) = \{(v_i, u_j) \mid 1 \leq i \leq m+1, 1 \leq j \leq n+1\}$ clearly $K_{1,m}$ and $K_{1,n}$ has the non increasing order of degree sequence $(m, \underbrace{1, 1, \dots, 1}_{m \text{ times}})$ and $(n, \underbrace{1, 1, \dots, 1}_{n \text{ times}})$ respectively.

Then the degree sequence of $(K_{1,m} \square K_{1,n})$ is

$$(m+n, \underbrace{m+1, m+1, \dots, m+1}_{n \text{ times}}, \underbrace{n+1, n+1, \dots, n+1}_{m \text{ times}}, \underbrace{2, 2, \dots, 2}_{mn \text{ times}})$$

Let $D_1 = \{(v_i, u_j) \mid 2 \leq i \leq m+1, 2 \leq j \leq n+1\}$

Then, clearly D_1 weakly dominate all other vertices of $V(K_{1,m} \square K_{1,n})$ except (v_1, u_1) . For chromaticity, we have to choose a unique vertex say (x, y) from the set $\{(u_1, v_j) \mid 2 \leq j \leq m+1\} \cup \{(u_i, v_1) \mid 2 \leq i \leq n+1\}$ which is adjacent to a vertex in D_1 , since $\chi(K_{1,m} \square K_{1,n}) = 2$.

Let $D = D_1 \cup \{(x, y)\}$ then D is the chromatic weak dominating set of $(K_{1,m} \square K_{1,n})$. Therefore $\gamma_w^c(K_{1,m} \square K_{1,n}) \leq |D| + 1 = mn + 1$. Therefore $\gamma_w^c(K_{1,m} \square K_{1,n}) \leq mn + 1$.

Suppose $\gamma_w^c(K_{1,m} \square K_{1,n}) < |D|$. clearly $\chi(< D >) = 2$. If we remove a vertex v from D_1 then there exist no vertex u such that v can weakly dominated by v . Also, suppose we remove the vertex (x, y) , then $\chi(< D >) = 1 < 2 = \chi(G)$. Hence $\gamma_w^c(K_{1,m} \square K_{1,n}) = |D|$.

Therefore $\gamma_w^c(K_{1,m} \square K_{1,n}) = mn + 1$.

Theorem 1.3 $\gamma_w^c(K_{1,3} \square P_n) = n + 4$.

Proof. Let P_n be a path with vertex set $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and let $\{u_1, u_2, u_3, u_4\}$ be the vertex set of $K_{1,3}$ with the full degree vertex u_1 .

Then $V(K_{1,3} \square P_n) = \{(u_i, v_j) \mid 1 \leq i \leq 4, 1 \leq j \leq n\}$

Let $D_1 = \{(u_2, v_1), (u_3, v_1), (u_4, v_1), (u_2, v_n), (u_3, v_n), (u_4, v_n)\}$

Let D_2 be set of all vertices choosed as follows. For each j , choose a vertex (u_i, v_j) such that $d(v_{j_1}, v_{j_2}) \leq 3$, for all $j_1, j_2 \in 1, 2, \dots, n$ and for $i = 1, 2, 3$. Clearly $D_1 \cup D_2$ is a chromatic weak dominating set of $K_{1,3} \square P_n$. Therefore $\gamma_w^c(K_{1,3} \square P_n) \leq |D_1 \cup D_2| = |D_1| + |D_2| = 6 + n - 2 = n + 4$. Suppose $\gamma_w^c(K_{1,3} \square P_n) < |D_1 \cup D_2|$. Clearly $\chi(< D_1 \cup D_2 >) = 2$. If we remove a vertex v from D_1 then there exist no vertex u such that v can weakly dominated by v .

Also, suppose we remove the vertex (u_i, v_j) either $i=1$ or $i=2$ or $i=3$ and $j=2$ to $n-1$ then

$(u_2, v_j) \in D_2$, since (u_2, v_j) cannot be dominated by the vertex of D_1 and D_2 . Since both cases leads to the contradiction

Hence $\gamma_w^c(K_{1,3} \square P_n) = |D_1 \cup D_2|$. Therefore $\gamma_w^c(K_{1,3} \square P_n) = n + 4$.

Theorem 1.4 $\gamma_w^c(P_3 \square P_n) = n + 2$.

Proof. Let P_n be a path with vertex set $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $V(P_3) = \{u_1, u_2, u_3\}$ Then $V(P_3 \square P_n) = \{(u_i, v_j) / 1 \leq i \leq 3, 1 \leq j \leq n\}$

Let $D_1 = \{(u_1, v_1), (u_1, v_n), (u_3, v_4), (u_3, v_n)\}$ and

$D_2 = \{(u_1, v_2), (u_3, v_3), (u_1, v_4), (u_3, v_5), \dots, (u_i, v_{n-1})\}$ either $i = 1$ or $i = 3$.

Clearly $D_1 \cup D_2$ is a chromatic weak dominating set of $P_3 \square P_n$. Therefore $\gamma_w^c(P_3 \square P_n) \leq |D_1 \cup D_2| = |D_1| + |D_2| = 4 + n - 2 = n + 2$. Suppose $\gamma_w^c(P_3 \square P_n) < |D_1 \cup D_2|$ Clearly $\chi(< D_1 \cup D_2 >) = 2$.

If we remove a vertex v from D_1 then there exist no vertex u such that v can weakly dominated by u . Also, suppose we remove the vertex (u_i, v_j) either $i=1$ or $i=3$ and $j=2$ to $n-1$ then $(u_2, v_j) \in D_2$, since (u_2, v_j) cannot be dominated by the vertex of D_1 and D_2 . Since both cases leads to the contradiction.

Hence $\gamma_w^c(P_3 \square P_n) = |D_1 \cup D_2|$. Therefore $\gamma_w^c(P_3 \square P_n) = n + 2$.

Remark 1.5 Let us consider the following grid subgraph H , of $P_n \square P_m$ with the vertex set $(x_i, y_j) / 1 \leq i \leq n, 1 \leq j \leq m$.

Let

$S_1^{(i)} =$

$(x_{i+1}, y_{j+1}), (x_{i+1}, y_{j+2})$ or $(x_{i+1}, y_{j+1}), (x_{i+2}, y_{j+1})$ or $(x_{i+2}, y_{j+1}), (x_{i+2}, y_{j+2})$ or

$(x_{i+2}, y_{j+2}), (x_{i+1}, y_{j+2})$ and $S_2(i) = (x_{i+1}, y_{i+1}), (x_{i+2}, y_{j+2})$ or $(x_{i+1}, y_{i+2}), (x_{i+2}, y_{j+2})$

Then clearly $S_1(i)$ is a Chromatic weak dominating set of H and $S_2(i)$ is a weak dominating set of H .

Theorem 1.6 For $n \geq 4$ and for $k \in N$ $\gamma_w^c(P_4 \square P_n) = \begin{cases} 4k + 2 & \text{if } n = 3k + 1 \\ 4k + 4 & \text{if } n = 3k + 2 \\ 4k + 5 & \text{if } n = 3k + 3 \end{cases}$

Proof. Let P_n be a path with vertex set $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $V(P_4) = \{u_1, u_2, u_3, u_4\}$ Then $V(P_4 \square P_n) = \{(u_i, v_j) / 1 \leq i \leq 4, 1 \leq j \leq n\}$.

Case 1: Let $n = 3k + 1$

Let $D_1 = \{(u_1, v_1), (u_1, v_n), (u_4, v_1), (u_4, v_n)\}$,

$D_2 = \{(u_1, v_4), (u_4, v_4), (u_1, v_7), (u_4, v_7), \dots, (u_1, v_{3i+1}), (u_4, v_{3i+1}), \dots, (u_1, v_n), (u_4, v_n)\}$

$D_3 = \cup S^{(i)}$ where $S^{(i)} = S_1^{(i)}$ or $S_2^{(i)}$ and $S^{(i)} = S_1^{(i)}$ for atleast one i .

Let $D = D_1 \cup D_2 \cup D_3$. Then, clearly D is a chromatic weak dominating set of $P_4 \square P_n$

Therefore $\gamma_w^c(P_4 \square P_n) \leq |D| = 4 + 2(k - 1) + 2k = 4 + 2k - 2 + 2k = 4k + 2$.

Case 2: Let $n = 3k + 2$.

Let $D_1 = \{(u_1, v_1), (u_1, v_n), (u_4, v_1), (u_4, v_n)\}$. For each $i = 1, 4$,

$$D_2 = \{(u_i, v_4), (u_i, v_7), \dots, (u_i, v_{3t+1}), (u_i, v_{3t+2}), \dots, (u_i, v_{n-3})\} \text{ for } t = 1, 2, \dots, k,$$

$$D_3 = \bigcup S^{(i)} \text{ where } S^{(i)} = S_1^{(i)} \text{ or } S_2^{(i)} \text{ and } S^{(i)} = S_1^{(i)} \text{ for atleast one } i.$$

Let $D = D_1 \cup D_2 \cup D_3$. Then, clearly D is a chromatic weak dominating set of $P_4 \square P_n$

$$\text{Therefore } \gamma_w^c(P_4 \square P_n) \leq |D| = 4 + 2k + 2k = 4 + 2k + 2k = 4k + 4.$$

Case 3: Let $n = 3k+3$

Let $D_1 = \{(u_1, v_1), (u-1, v_n), (u_4, v_1), (u_4, v_n)\}$. For each $i = 1, 4$, $D_2 = \{(u_i, v_4), (u_i, v_7), \dots, (u_i, v_{3t+1}), (u_i, v_{3t+3}), \dots, (u_i, v_{n-6}), (u_i, v_{n-3})\}$ for $t = 1, 2, \dots, k$. $D_3 = \bigcup S^{(i)}$ where $S^{(i)} = S_1^{(i)}$ or $S_2^{(i)}$ and $S^{(i)} = S_1^{(i)}$ for atleast one i . $D_4 = \{(u_2, v_{3t+2})\} \text{ or } \{(u_3, v_{3t+2})\} \text{ for } t = 1, 2, \dots, k$. Let $D = D_1 \cup D_2 \cup D_3 \cup D_4$. Then, clearly D is a chromatic weak dominating set of $P_4 \square P_n$. Therefore $\gamma_w^c(P_4 \square P_n) \leq |D| = 4 + 2k + 2k + 1 = 4k + 5$. Conversely, Suppose $\gamma_w^c(P_4 \square P_n) < |D|$ clearly $\chi(< D >) = 2$. If we remove a vertex v from D then there exist no vertex u such that v can weakly dominated by v . Also, suppose we remove the vertex (x, y) , then $\chi(< D >) = 1 < 2 = \chi(< G >)$.

$$\text{Hence } \gamma_w^c(P_4 \square P_n) = |D|. \quad \gamma_w^c(P_4 \square P_n) = \begin{cases} 4k + 2 & \text{if } n = 3k + 1 \\ 4k + 4 & \text{if } n = 3k + 2 \\ 4k + 5 & \text{if } n = 3k + 3 \end{cases}$$

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