Chromatic Strong Domatic Partition in Graphs

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Abstract

A vertex subset D is a chromatic strong dominating set if D is a strong dominating set and χ(< D >) = χ(G). The minimum cardinality of chromatic strong dominating set is called the chromatic strong domination number and is denoted by 𝛾𝑠𝑐(𝐺).

A chromatic strong domatic partition (csd-partition) of a graph G is a partition of V into chromatic strong dominating sets. The maximum cardinality of a partition of V into chromatic strong dominating sets is the csd-partition number and is denoted by 𝑑𝑠𝑐(𝐺).

AMS Subject Classification: 05C69
Key words: Domination, Strong domination, Chromatic Number, Domatic Number.

1. Introduction

All graphs considered here are finite, undirected and simple. Let $G = (V, E)$ be a graph of order n and size m. A subset D of V is called a dominating set of G if every vertex in V - D is adjacent to some vertex in D. Dominating sets were first studied by Berge [3] and Ore [8].

The domination number of a graph G is the minimum cardinality of a dominating set of G. The domatic number $d(G)$ of a graph $G = (V, E)$ is the maximum positive integer k such that V can be partition into k pairwise disjoint dominating sets $D_1, D_2, ..., D_k$. A partition of V into pairwise disjoint dominating sets is called a domatic partition. The concept of a domatic number was introduced in [5]. The word ‘domatic’ was created from the words ‘dominating’ and ‘chromatic’, in the same way the word ‘smog’ was created from the words ‘smoke’ and ‘fog’. In a certain sense, a domatic number is analogous to the chromatic number of a graph, which is the minimum positive integer k such that the vertex set can be partitioned into k pairwise disjoint stable sets. For $D \subseteq V$, the subgraph induced by D is denoted by $< D >$. Prof. E. Samapathkumar and L. Pushpalatha introduced the concept of strong domination in graphs. A set $D \subseteq V$ is called a strong dominating set if for every vertex $v \in V - D$, there exists a vertex $u \in D$ such that $uv \in E$ and $\deg(u) = \deg(v)$. The minimum cardinality of a strong dominating set is called the strong domination number of G and is denoted by $\gamma_s(G)$. The strong dominating set of cardinality $\gamma_s(G)$ is denoted by $\gamma_s$-set of G. A coloring of a graph G is an assignment of colors to all its vertices such that all pairs of adjacent vertices are assigned different colors. The chromatic number $\chi(G)$ is the minimum number of colors necessary to a coloring of G. A set $D \subseteq V$ is a chromatic strong dominating set if D is a strong dominating set and $\chi(< D >) = \chi(G)$. The minimum cardinality of a chromatic strong dominating set is called the chromatic strong domination number of G and...
is denoted by $\gamma_s^c(G)$.

2 Chromatic Strong Domatic Partition Sets

Definition 2.1 A chromatic strong domatic partition (csd-partition) of graph $G$ is a partition of $V$ into chromatic strong dominating sets. The maximum cardinality of a partition of $V$ into chromatic strong dominating sets is the chromatic strong domatic partition number and is denoted by $d_{sc}^s(G)$.

Observation 2.2
1. For any graph $G$, $d_{sc}^s(G) \leq d_{ch}(G) \leq d(G) \leq \delta(G) + 1$
2. If a graph $G$ has a pendant vertex, then $d_{sc}^s(G) \leq 2$
3. There exist graphs for which $d_{sc}^s(G) = \delta(G) + 1$. For example, $d_{sc}^s(P_8) = 2$ and $\delta(P_8) = 1$.

Definition 2.3 A graph $G$ is said to be chromatic strong domatically full if $d_{sc}^s(G) = \delta(G) + 1$

Proposition 2.4 For any graph $G$, $d_{sc}^s(G) \leq \frac{n}{\gamma_s^c(G)}$

Proof: Let $\{V_1, V_2, \ldots, V_k\}$ be a maximum chromatic strong partition of $V$. Therefore $k = d_{sc}^s(G)$. Since $V_i$ is a chromatic strong dominating set $|V_i| \geq \gamma_s^c(G), 1 \leq i \leq k$. Since $V = \bigcup_{i=1}^{k} V_i$, $n = \sum_{i=1}^{k} |V_i| \geq \sum_{i=1}^{k} \gamma_s^c(G), n \geq k \cdot \gamma_s^c(G)$. Therefore $k \leq \frac{n}{\gamma_s^c(G)}$. Therefore $d_{sc}^s(G) \leq \frac{n}{\gamma_s^c(G)}$

Proposition 2.5
1. If $\gamma_s^c(G) > \frac{n}{2}$, then $d_{sc}^s(G) = 1$
2. If $G$ has a strong isolate vertex, then $d_{sc}^s(G) = 1$
3. If $G$ has unique maximum clique, then $d_{sc}^s(G) = 1$
4. If $G$ is $\chi$-critical, then $d_{sc}^s(G) = 1$

Proposition 2.6
1. $d_{sc}^s(K_n) = 1$
2. $d_{sc}^s(K_{1,n-1}) = 1$
3. $d_{sc}^s(F_n) = 1$
4. $d_{sc}^s(D_{r,s}) = 1$
5. $d_{sc}^s(K^+_n) = 1$
6. $d_{sc}^s(W_n) = 1$
7. \( d_s^c(K_{m,n}) = \begin{cases} 
1 & \text{if } m \neq n \\
\frac{m}{2} & \text{if } m = n 
\end{cases} \)

8. \( d_s^c(C_n) = \begin{cases} 
2 & \text{if } n \text{ is even} \\
1 & \text{if } n \text{ is odd} 
\end{cases} \)

9. \( d_s^c(P_n) = \begin{cases} 
2 & \text{if } n \geq 8 \\
1 & \text{if } 2 \leq n \leq 7 
\end{cases} \)

**Proposition 2.7** 

G is non trivial if and only if \( \gamma_s^c(G) \geq 2 \)

**Theorem 2.8** 

For any non trivial graph \( G \), \( d_s^c(G) \leq \frac{n}{2} \)

**Proof:** It follows proposition 2.4 and 2.7

**Theorem 2.9** 

For any graph \( G \) with even order \( n \), \( d_s^c(G) = \frac{n}{2} \) if and only if \( G = K_{n,n} \) or \( K_2 \)

**Proof:** If \( G = K_1 \) then \( d_s^c(G) = 1 = n \neq \frac{n}{2} \). Therefore \( G \neq K_1 \). Let \( G \neq K_2 \) and \( d_s^c(G) = \frac{n}{2} \). Let \( V_1, V_2, V_3, \ldots, V_n \) be a \( csd \)-partition of \( G \). Then \( |V_i| \leq 2 \) for all \( i \). Since \( n \geq 2 \), \( |V_i| \geq 2 \) for all \( i \). Therefore \( |V_i| = 1 \Rightarrow G = K_1 \). Therefore \( |V_i| = 2 \) for all \( i \). If \( V_i \) is independent for some \( i \), then \( \chi(G) = \chi(<V_i>) = 1 \). Hence, \( G = K_{n,n} \) and \( d_s^c(K_{n,n}) = \frac{n}{2} \). Thus, \( G = K_2 \), which is a contradiction to \( G \neq K_2 \). Therefore \( V_i \) is not independent for every \( i \).

Therefore, \( \chi(G) = \chi(<V_i>) = 2 \). Therefore \( G \) is nontrivial bipartite. Let \( X, Y \) be the bipartition of \( G \). Let \( X \cap V_i = \{x_i\} \) and \( Y \cap V_i = \{y_i\} \). Since \( V_1, V_2, V_3, \ldots, V_n \) is a partition of \( V \), \( |X| = |Y| = \frac{n}{2} \). Since \( V_i = \{x_i, y_i\} \) is a dominating set and \( X, Y \) are independent sets, each \( y_j \) is adjacent to \( x_i \) and each \( x_j \) is adjacent to \( y_i \). Since \( i \) is arbitrary, \( G \) is complete bipartite graph. Thus, \( G = K_{\frac{n}{2}, \frac{n}{2}} \)

**Theorem 2.10** 

If a graph \( G \) has \( d_s^c(G) \geq 2 \), then \( \gamma_s^c(G) + d_s^c(G) \leq \left[ \frac{n}{2} \right] + 2 \)

**Proof:** Let \( G \) be a graph with \( d_s^c(G) \geq 2 \). Then \( \gamma_s^c(G) \leq \left[ \frac{n}{2} \right] \). Since \( G \neq K_1, \gamma_s^c(G) \geq 2 \), and so \( d_s^c(G) \leq \left[ \frac{n}{2} \right] \). If either \( \gamma_s^c(G) = 2 \) or \( d_s^c(G) = 2 \), then the bound obviously holds. If \( \gamma_s^c(G) \geq 4 \) and \( d_s^c(G) \geq 4 \), then since \( \gamma_s^c(G) d_s^c(G) \leq n \), we have \( \gamma_s^c(G) \leq \left[ \frac{n}{d_s^c(G)} \right] \) and \( d_s^c(G) \leq \left[ \frac{n}{\gamma_s^c(G)} \right] \). That is, \( \gamma_s^c(G) \leq \left[ \frac{n}{4} \right] \). Hence, \( \gamma_s^c(G) + d_s^c(G) \leq 2 \left[ \frac{n}{4} \right] < \left[ \frac{n}{2} \right] + 2 \). Let \( d_s^c(G) = 3 \) or \( \gamma_s^c(G) = 3 \). Then \( \gamma_s^c(G) + d_s^c(G) \leq 3 + \left[ \frac{n}{3} \right] \). Since
\[ 3 = d_\gamma'(G) \text{ or } \gamma'(G) \leq \left\lfloor \frac{n}{2} \right\rfloor \text{ we have } n \geq 6. \text{ For } n \geq 6, \quad 3 + \left\lfloor \frac{n}{2} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor + 2. \text{ Therefore } \gamma'(G) + d_\gamma'(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 2. \]

**Theorem 2.11** For any graph \( G \), \( 4 \leq d_\gamma'(G), d_\gamma(G) \leq \frac{n^2}{4} \) with equality if and only if \( G = K_2 \).

**Proof:** Since \( n > 1 \) both \( G \) and \( \overline{G} \) having chromatic strong domatic number at least 2. Thus, \( 4 \leq d_\gamma'(G), d_\gamma(G) \). This lower bound is sharp may be seen by taking \( G = K_2 \). Since \( d_\gamma'(G) \leq \frac{n}{2} \) and \( d_\gamma(G) \), we get the upper bound if \( n > 1 \). \( d_\gamma'(G), d_\gamma(G) = \frac{n^2}{4} \) if and only if \( d_\gamma'(G) = \frac{n}{2} \) and \( d_\gamma(G) = \frac{n}{2} \). That is if and only if \( G = K_{\frac{n}{2}} \) or \( \overline{K_{\frac{n}{2}}} \) and \( d_\gamma'(G) = 1 \). Therefore \( d_\gamma'(G), d_\gamma(G) = \frac{n}{2} = \frac{n^2}{4} \) if and only if \( n = 2 \). that is \( G = K_2 \). Let \( G = K_{\frac{n}{2}} \). Then \( \overline{G} = K_2 = K_{\frac{n}{2}} \). Therefore \( d_\gamma'(G), d_\gamma(G) = \frac{n^2}{4} \) if and only if \( \overline{K_{\frac{n}{2}}} \).

**Theorem 2.12** For any graph \( G \), \( d_\gamma'(G) + d_\gamma(G) \leq n \), with equality if and only if \( G = K_2 \) or \( \overline{K_2} \).

**Proof:** Let \( n \geq 2 \). \( d_\gamma'(G) + d_\gamma(G) \leq \frac{n}{2} + \frac{n}{2} = n \). \( d_\gamma'(G) + d_\gamma(G) = n \) if and only if \( d_\gamma'(G) = \frac{n}{2} = d_\gamma(G) \). That is if and only if \( G = K_{\frac{n}{2}} \) or \( \overline{K_2} \) and \( \overline{G} = K_{\frac{n}{2}} \) or \( \overline{K_2} \). Let \( G = K_{\frac{n}{2}} \) or \( \overline{K_2} \) or \( \overline{K_2} \). In the case, \( d_\gamma'(G) = \frac{n}{2} \) and \( d_\gamma(G) = 1 \). Therefore \( d_\gamma'(G) + d_\gamma(G) = \frac{n}{2} + 1 = n \) if and only if \( n = 2 \). Therefore \( G = K_2 \) and \( \overline{G} = K_{\frac{n}{2}} \). If \( G = \overline{K_2} \), then \( \overline{G} = K_2 = K_{\frac{n}{2}} \). Then also \( d_\gamma'(G) + d_\gamma(G) = 2 = n \).

**Theorem 2.13** Let \( G \) be a graph such that \( G \) and \( \overline{G} \) are not chromatically strong domatically full. Then \( d_\gamma'(G) + d_\gamma(G) \leq n - 1 \).

**Proof:** \( d_\gamma'(G) \leq \delta(G) \) and \( d_\gamma(G) \leq \delta(\overline{G}) \). Therefore \( d_\gamma'(G) + d_\gamma(G) \leq \delta(G) + \delta(\overline{G}) = n - 1 \).

**Example 2.14** If \( G = C_4 \), then \( G \) is not domatically full and \( \overline{G} = 2K_2 \) is also not domatically full. Here, \( d_\gamma'(G) + d_\gamma(G) = 2 + 1 = 3 = 4 - 1 = n - 1 \).

**Corollary 2.15** Under the hypothesis of the Theorem, if equality holds, then \( G \) is regular.

**Proof:** Suppose \( G \) is not regular. Then \( \delta(G) \not< \Delta(G) \). \( d_\gamma'(G) + d_\gamma(G) \leq \delta(G) + \delta(\overline{G}) = n + 1 \), a contradiction. Therefore \( G \) is regular.

**Theorem 2.16** Let \( G \) be a graph such that \( G \) and \( \overline{G} \) are chromatically strong domatically full. Let \( \delta(G) \geq 1 \) and \( \Delta(G) \leq n - 2 \). Then \( d_\gamma'(G) + d_\gamma(G) = n - 1 \) if and only if \( G \) are \( \overline{G} \) is \( C_4 \).

**Proof:** Proceeding as in ?? [Hedetniemi Book]. We get the result.

**Theorem 2.17** For any graph \( G \), \( \gamma'_\gamma(G) + d_\gamma(G) \leq n + 1 \).

**Proof:** Suppose \( n = 1 \). Then \( G = K_1 \). \( \gamma'_\gamma(G) = 1 \). \( d_\gamma(G) = 1 \). Therefore \( \gamma'_\gamma(G) + d_\gamma(G) = 2 = n + 1 \). Let \( n > 1 \). Suppose \( \gamma'_\gamma(G) = n \). Then \( d_\gamma(G) = 1 \). Therefore \( \gamma'_\gamma(G) + d_\gamma(G) = n + 1 \). Suppose \( \gamma'_\gamma(G) < n \). That is \( \gamma'_\gamma(G) \leq n - 1 \).
Case: $\gamma_s^c(G) \leq \frac{n}{2}$. Since $n > 1$, $d_s^c(G) \leq \frac{n}{2}$. Therefore $\gamma_s^c(G) + d_s^c(G) \leq n < n + 1$. Case: $\gamma_s^c(G) \leq \frac{n}{2}$.

Since $\gamma_s^c(G) \leq n$, $d_s^c(G) \leq \frac{n}{2} = 2$. Therefore $\gamma_s^c(G) + d_s^c(G) \leq n - 1 + 2 = n + 1$. Hence the theorem.

**Theorem 2.18** Let $G$ be a graph without strong vertices. Then $\gamma_s^c(G) + d_s^c(G) = n + 1$ if and only if

(i). $G$ is connected and $G$ is $\chi$-critical

(ii). $G$ is disconnected and either $G = K_n$ or $G$ has exactly one non trivial component which is $\chi$-critical.

**Proof:** Let $n = 1$. Then $\gamma_s^c(G) + d_s^c(G) = 2$ when $G = K_1$. $\gamma_s^c(G) + d_s^c(G) = n + 1$ if and only if either $\gamma_s^c(G) = n$, $d_s^c(G) = 1$ or $\gamma_s^c(G) = n - 1$ and $d_s^c(G) = 2$ when $\gamma_s^c(G) = n - 1$, then $d_s^c(G) \leq \frac{n}{n - 1} < 1 + \frac{1}{n - 1} < 2$. $\gamma_s^c(G) + d_s^c(G) < n + 1$, a contradiction. Therefore $\gamma_s^c(G) + d_s^c(G) = n + 1$ if and only if $\gamma_s^c(G) = n$, $d_s^c(G) = 1$. That is if and only if $G$ is $\chi$-critical and connected or $G$ is disconnected and $G = K_n$ or $G$ has exactly one non-trivial component which is $\chi$-critical.

**References**


