Some Classes On \( \alpha\)-Para Kenmotsu Manifolds

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Abstract:

The object of this paper is to study Ricci pseudo-symmetric and Ricci generalized pseudo symmetric \( \alpha\)-Para Kenmotsu manifolds. We extend our study to \( \phi\)-pseudo symmetric and \( \phi\)-pseudo Ricci symmetric \( \alpha\)-Para Kenmotsu manifolds. Futher we obtain some results on semi generalized \( \phi\)-recurrent and semi generalized projectively \( \phi\)-recurrent \( \alpha\)-Para Kenmotsu manifolds.

keywords: Ricci generalized pseudo symmetric, Semi generalized \( \phi\)-recurrent, Einstein manifold, \( \eta\)-Einstein, \( \phi\)-pseudo symmetric, \( \phi\)-pseudo Ricci symmetric, \( \alpha\)-Para Kenmotsu manifolds.

1. Introduction

In 1972, K.Kenmotsu [17] introduced Kenmotsu manifolds and the geometry of almost Kenmotsu manifolds have been investigated in many aspects [4]–[6]. Most of the results contained in [4]–[5] can be easily generalized to the class of almost \( \alpha\)-Kenmotsu manifolds, where \( \alpha\) is a non-zero real number [6].

Kaneyuki and Williams [25] introduced almost paracontact geometry in 1985. After that many authors have been continued the same. A systematic study of almost paracontact metric manifolds was carried out by Zamkovoy [26]. However such structures were also studied by Buchner and Rosca [14],[15]. [16] Rossca and Vanhecke [24]. The curvature identities for different classes of almost paracontact metric manifolds were obtained in [23]. Further, almost para-Hermitian structures on the tangent bundle of an almost para- co Hermitian manifolds were studied by Bejan [3]. A class of \( \alpha\)-para Kenmotsu manifolds were studied by K.Srivastava and S.K.Srivastava [18] and \( \xi\)-conformally flat contact metric manifolds were studied by Zhen et al. [7].

In the differential geometric point of view projective curvature tensor considered as an important tensor. Let \( M\) be a \((2n+1)\)-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of \( M\) and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then \( M\) is said to be locally projectively flat. For \( n \geq 1\), \( M\) is locally projectively flat if and only if the well-known projective curvature tensor \( P\) vanishes, where \( P\) is defined by [8]

\[ P(X, Y, Z) = R(X, Y)Z - \frac{2n}{S(Y, Z)X - S(X, Z)Y}. \] (1.1)

for all \( X, Y, Z \in \mathcal{T}(M)\), where \( R\) is the curvature tensor and \( S\) is the Ricci tensor.

For a \((0, k)\)-tensor field \( T, k \geq 1\), on \((M^n, g)\) we define the tensor \( R \cdot T\) and \( Q(g, T)\) by

\[ (R(X, Y) \cdot T)(X_1, X_2, ..., X_k) = -T(R(X, Y)X_1, X_2, ..., X_k) - T(X_1, R(X, Y)X_2, ..., X_k) - T(X_1, X_2, ..., R(X, Y)X_k). \] (1.2)

And

\[ Q(g, T)(X_1, X_2, ..., X_k) = -T((X \wedge Y)X_1, X_2, ..., X_k) - T(X_1, (X \wedge Y)X_2, ..., X_k) - T(X_1, X_2, ..., (X \wedge Y)X_k). \] (1.3)

We define endomorphisms \( \mathcal{A}_B Y\) by

\[ (X \wedge_B Y)Z = B(Y, Z)X - B(X, Z)Y. \] (1.4)

If the tensor \( R \cdot S\) and \( Q(g, S)\) are linearly dependent then \( M^n\) is called Ricci pseudo-symmetric [21]. This is equivalent to

\[ R \cdot S = fQ(g, S). \] (1.5)

for the set \( U_{f} = \{ x \in M : S = 0 \text{ at } x \} \), where \( f\) is some function on \( U_{f}\). Analogously, if the tensors \( R \cdot R\) and \( Q(S, R)\) are linearly dependent then \( M^n\) is called Ricci generalized pseudo-symmetric [21]. This is equivalent to

\[ R \cdot R = fQ(S, R). \] (1.6)

for the set \( U_{f} = \{ x \in M : R = 0 \text{ at } x \} \), where \( f\) is some function on \( U_{f}\). A very important subclass of this class of manifolds realizing the condition is

\[ R \cdot R = Q(S, R). \]

Further we define the tensors \( R \cdot R\) and \( R \cdot S\) on \((M^n, g)\) by


And

\[ -R(U, R(X, Y)V)W = -R(U, R(X, Y)V)W. \] (1.7)

\[ (R(X, Y) \cdot S)(U, V) = -S(R(X, Y)U, V) - S(U, R(X, Y)V). \] (1.8)

where \( X, Y, Z \in \mathfrak{g}(M)\), the set of all differentiable vector fields on \( M\), \( B\) is the symmetric (0,2)-tensor, \( R\) is the Riemannian curvature tensor of type \((1, 3)\) and \( \nabla\) is the Levi-Civita connection.

Motivated by the above results we studied, the Ricci pseudo
symmetric α-para Kenmotsu manifolds in section 3 and in the section 4, we study Ricci generalized pseudo-symmetric α-para Kenmotsu manifolds. Section 5 is devoted to the study of pseudo-symmetric α-para Kenmotsu manifolds. In section 6, we establish pseudo Ricci symmetric α-para Kenmotsu manifolds. In section 7, we discuss the semi generalized ϕ-recurrent α-para Kenmotsu manifolds. In section 8, we characterize generalized projectively ϕ-recurrent α-para Kenmotsu manifolds.

2. Preliminaries

A smooth manifold M of dimension (2n + 1) is called an almost paracomplex manifold ([22],[9]) equipped with the structure (φ, ξ, η) where φ is a tensor field of type (1,1), a vector field ξ and a 1-form η satisfying

\[ \phi^2 = 1 - \eta \otimes \xi, \] (2.1)

\[ \eta(\xi) = 1, \] (2.2)

\[ \phi^2 \eta = 0, \eta \circ \phi = 0, \text{rank}(\eta) = 2n. \] (2.3)

If an almost paracomplex manifold M admits a pseudo-Riemannian metric g satisfying

\[ \eta(X, \phi Y) + \eta(Y, \phi X) = 0, \]

\[ g(\phi X, \phi Y) = -g(X, Y) + \eta(X) \eta(Y), \] (2.4)

\[ g(X, \phi Y) = g(\phi X, Y), \] (2.5)

with signature of g as (n + 1, n) for any vector fields X, Y \in \chi(M), then the manifold M is called almost paracomplex metric manifold.

An almost paracomplex structure is said to be a contact structure if \( g(\phi X, Y) = \eta(X) \) with the associated metric g [26]. For an almost paracomplex manifold, there always exists a special kind of local pseudo orthonormal φ basis \{X_1, \phi X_1, \xi, \eta(X_1)\}, \{X_1\}'s and \{\xi, \eta(X_1)\}'s are space-like and time-like. Thus, an almost paracomplex metric manifold is an odd dimensional manifold.

An almost paracomplex metric manifold is said to be normal if the induced almost paracomplex structure J on the product manifold M^{2n+1} \times R, defined by d

\[ J X, f = df \]

\[ \phi X + \phi_f \eta(X) dt, \] (2.7)

is integrable where X is tangent to M, t is the coordinate of R and f is a smooth function on M^{2n+1} \times R. The condition for the space M to be normal is equivalent to vanishing of the \( 1, 2 \)-type torsion tensor \( N_\phi \) defined by \( N_\phi = [\phi, \phi] - 2d\eta \xi, \) if \([\phi, \phi] = \text{Nijenhuis torsion of } \phi. \)

For a 3-dimensional almost paracomplex metric manifold M, the following conditions are mutually equivalent[13]:

- M is normal,
- there exist differential functions α, β on M such that

\[ \langle \nabla_X \phi Y = \beta g(X, Y) \xi - \eta[Y, X] + \alpha g(\phi X, Y) \xi - \eta(Y) \phi X \rangle \]

- there exist differential function α, β on M such that

\[ \langle \nabla X \xi = \alpha \eta(Y, \xi) + \beta \phi Y \rangle, \]

where \( \nabla \) is the Levi-Civita connection of the pseudo-Riemannian metric g and α, β are given by

\[ 2\alpha = \text{Trace}[X \rightarrow \phi \xi], 2\beta = \text{Trace}[X \rightarrow \eta(X) \xi]. \]

A 3-dimensional normal almost paracomplex metric manifold M is said to be

Paracosymplectic if \( \alpha = \beta = 0 [23], \)

- \( \alpha \)-para Kenmotsu if \( \alpha \) is a non zero constant and \( \beta = 0 [12], \)
- quasi-parasasakian if and only if \( \alpha = 0 \) and \( \beta = 0 [27], \)
- \( \beta \)-para sasakian if and only if \( \alpha = 0 \) and \( \beta \) is a non zero constant, in particular para sasakian if \( \beta = -1 [26]. \)

In a 3-dimensional \( \alpha \)-para Kenmotsu manifold, the following results hold [18]:

\[ R(X, Y)Z \Rightarrow E + 2a^2 [\eta(Y, Z)g(X, Z)Y \Rightarrow E + 3a^2 \]

\[ 2[\eta(Y, Z)\eta(X) - \eta(X, \eta(Z))] \xi, \] (2.8)

\[ S(X, Y) \Rightarrow E + a^2 g(X, Y) \Rightarrow E + 3a^2 \eta(X) \eta(Y), \] (2.9)

\[ S(X, \xi) = -2a^2 \eta(X), \] (2.10)

\[ R(X, Y) \xi \Rightarrow -a^2 (\eta(Y) \xi - \eta(X) Y), \] (2.11)

\[ \langle \nabla_X \phi Y = \alpha g(\phi X, Y) \xi - \eta(Y) \phi X, \rangle, \] (2.12)

\[ \langle \nabla X \phi Y = \alpha g(\phi X, Y) \xi - \eta(Y) \phi X, \rangle, \] (2.13)

\[ \langle \nabla X \xi = \alpha \eta(X) \xi, \rangle \] (2.14)

for all vector fields \( X, Y, \phi X, JX \) and \( W \in \chi(M). \)

where \( r \) is the scalar curvature of the manifold and g is pseudo-metric.

3. Ricci Pseudo Symmetric A-Para Kenmotsu Manifolds

In this section we study Ricci pseudo-symmetric manifold, that
is, the manifold satisfying the condition \( R \cdot S = f Q(g, S) \). Assume that \( M \) is a Ricci pseudo-symmetric \( \alpha \)-para Kenmotsu manifold and \( X, Y, U, V \in \mathfrak{X}(M) \). We have from (1.5)

\[
(R(X, Y) - S(U, V) = f Q(g, S)(X, Y; U, V). \tag{3.1}
\]

It is equivalent to

\[
(R(X, Y) - S(U, V) = f(X \omega Y) \cdot S(U, V). \tag{3.2}
\]

From (1.8) and (1.3)

we have

\[
\begin{align*}
- S(R(X, Y) U, V) & = S(U, R(X, Y) V) = f [ - S(X \omega Y) U, V] - S(U, X \omega Y) V) ] \tag{3.3}
\end{align*}
\]

Using (1.4) we obtain, from (3.3), that

\[
- S(R(X, Y) U, V) - S(U, R(X, Y) V) = f [ - g(Y, U) S(X, V) + g(X, U) S(Y, V)]
\]

Substituting \( X = U = \xi \) in (3.4) and using (2.8) and (2.10) to obtain (3.4)

By above discussion we have the following:

\[
- a^2 + f S(Y, V) - 2a^2 g(Y, V) = 0. \tag{3.5}
\]

Then either \( f = -a^2 \) or, the manifold is an Einstein manifold of the form

\[
S(Y, V) = 2a^2 g(Y, V). \tag{3.6}
\]

Proposition 1. Every 3-dimensional Ricci pseudo-symmetric \( \alpha \)-para Kenmotsu manifold is of the form \( R \cdot S = f Q(g, S) \), provided the manifold is non-Einstein.

On the other hand, if the manifold is an Einstein manifold of the form (3.6), then it is clear that \( R \cdot S = f Q(g, S) \).

This leads the following Theorem 3.1. A 3-dimensional \( \alpha \)-para Kenmotsu manifold is Ricci pseudo-symmetric if and only if the manifold is an Einstein manifold provided \( f = -a^2 \).

In particular, if we consider \( Q(g, S) = 0 \), then by the similar argument of theorem 3.1 we can state the following:

Corollary 3.1. A 3-dimensional \( \alpha \)-para Kenmotsu manifold satisfies the condition

\[
Q(g, S) = 0 \text{ if and only if the manifold is an Einstein.}
\]

4. Ricci Generalized Pseudo-Symmetric \( \alpha \)-Para Kenmotsu Manifolds

In this section we deal with Ricci generalized pseudo-symmetric \( \alpha \)-para Kenmotsu manifolds. Let us suppose that \( M \) be an \( n \)-dimensional Ricci generalized pseudo-symmetric \( \alpha \)-para Kenmotsu manifolds. Then from (1.6) we have

\[
R \cdot R = f Q(S, R). \tag{4.1}
\]

which implies that

\[
(R(X, Y) \cdot R(U, V) = f (X \omega Y) \cdot R(U, V) + W \cdot R(X, Y) W)
\]

Using (1.7) and (1.3) we get, from (4.2), that

\[
R(X, Y) \cdot R(U, V) = f (X \omega Y) \cdot R(U, V) + W \cdot R(X, Y) W)
\]

Then either \( f = 1/3 \) or, the manifold is an Einstein manifold of the form

\[
- \alpha^2 + \frac{3}{2} f g(Y, V) \leq 0. \tag{4.8}
\]

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\[
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\]

The above equation yields

\[
- \alpha^2 + \frac{3}{2} f g(Y, V) \leq 0. \tag{4.8}
\]

Then either \( f = 1/3 \) or, the manifold is an Einstein manifold of the form

\[
- \alpha^2 + \frac{3}{2} f g(Y, V) \leq 0. \tag{4.8}
\]

The above equation yields

\[
- \alpha^2 + \frac{3}{2} f g(Y, V) \leq 0. \tag{4.8}
\]
S(Y, Z) = \(-2\alpha^2 g(Y, Z)\). (4.9) This leads to the following:

**Theorem 4.2.** A 3-dimensional Ricci generalized pseudo-symmetric \(\alpha\)-para Kenmotsu manifold is Einstein provided \(f = 0\).

By the above discussion we also have the following:

**Proposition 2.** Every 3-dimensional Ricci generalized pseudo-symmetric \(\alpha\)-para Kenmotsu manifold is of the form \(R \cdot R - R = \lambda Q(S, R)\), provided the manifold is non-Einstein.

In particular, if we consider \(Q(S, R) = 0\), then by the same argument of Theorem 4.2 we can state the following:

**Corollary 4.2.** If a 3-dimensional \(\alpha\)-para Kenmotsu manifold satisfies the condition

\[ Q(S, R) = 0 \]

then the manifold is an Einstein one.

**Corollary 4.3.** If a \(\alpha\)-para Kenmotsu manifold satisfies the condition \(R \cdot R = Q(S, R)\), then the manifold is an Einstein manifold.

5. \(\phi\)-**PSEUDO SYMMETRIC \(\alpha\)-PARA KENMOTSU MANIFOLDS**

**Definition 5.1.** [10] A \(\alpha\)-para Kenmotsu manifold \(M^3(\phi, \xi, \eta, \xi)\) is said to be a \(\phi\)-pseudo symmetric if the curvature tensor \(R\) satisfies

\[ \phi^2(\Phi W R)(X, Y | Z) = 2a(uW)R(X, Y | Z) + u(X)\mu(R(W, Y)Z) + u(Y)\mu(R(W, Y)X) + u(Z)\mu(R(W, Y)Y) + g(R(X, Y | Z), W) \]  

for any vector field \(X, Y, Z\) and \(W\), where \(u\) is a non-zero 1-form. In particular, if \(u = 0\), then the manifold is said to be \(\phi\)-symmetric [28].

We now consider a \(\alpha\)-para Kenmotsu manifold \((M^3, \phi, \xi)\), which is \(\phi\)-pseudo symmetric. Then by virtue of (2.1), it follows from (5.1), that

\[ (\Phi W R)(X, Y | Z) = 2a(uW)R(X, Y | Z) + u(X)\mu(R(W, Y)Z) + u(Y)\mu(R(W, Y)X) + u(Z)\mu(R(W, Y)Y) + g(R(X, Y | Z), W) \]  

for which we have,

\[ g(\Phi W R)(X, Y | Z, U) = 2a(uW)(g(R(X, Y | Z), U) + u(X)\xi g(R(W, Y)Z, U) + u(Y)\xi g(R(W, Y)X, U) + u(Z)\xi g(R(W, Y)Y, U) + g(R(X, Y | Z), W)u(U)). \]  

Taking an orthonormal frame field and then contracting (5.3) over \(X\) and \(U\) then using (2.1), we get

\[ (\Phi W R)(Y, Z) = 2a(uW)\mu(X, Y | Z) + u(Y)\mu(R(X, Y | Z), W) + u(U)\mu(R(X, Y | Z), W) \]  

and

\[ (\phi W R)(X, Y | Z) = 0. \]  

By virtue of (5.5) it follows, from (5.4), that

\[ (\phi W R)(X, Y | Z) = 2a(uW)\mu(X, Y | Z) + u(Y)\mu(R(X, Y | Z), W) + u(U)\mu(R(X, Y | Z), W) \]  

This leads to the following:

**Theorem 5.3.** A \(\phi\)-pseudo symmetric \(\alpha\)-para Kenmotsu manifold is pseudo Ricci symmetric if and only if \(u(R(W, Y | Z), W) = 0\), for any vector field \(W, Y, Z\).

Setting \(Z = \xi\) in (5.2) and using (2.8) and (2.11), we get

\[ (u(\xi) + a)\mu(R(X, Y | W), W) = \alpha B(X, Y) \]  

and

\[ = 2u(W)(g(R(X, Y | Z), W) + u(X)\mu(R(W, Y)Z) + u(Y)\mu(R(W, Y)X) + u(Z)\mu(R(W, Y)Y) + g(R(X, Y | Z), W)u(U)) \]  

Using the relation \(g(\phi W R)(X, Y | Z, U) = -g(\phi W R)(X, Y | Z, U) \) and (2.6), (2.8), we have

\[ g(\phi W R)(X, Y | Z, U) = 0. \]  

Changing \(W\) by \(\phi W\), we obtain

\[ (u(\xi) + a)\mu(g(R(X, Y | W), W), U) = \alpha B(X, Y) \]  

and

\[ + \alpha B(X, Y)R(g(R(X, Y | W), W), U) + u(Y)\mu(R(X, Y | Z), W) \]  

setting \(X = U = \xi\) in (5.8) and taking summation over \(i = 1, 2, \ldots, n\), we get

\[ (u(\xi) + a)\mu(S(Y, W), W) = \alpha B(Y, W) \]  

Replacing \(W\) by \(\phi W\) in (5.9)

\[ (u(\xi) + a)\mu(S(Y, W), W) = -\alpha B(Y, W) \]  

we state

**Theorem 5.4.** In a \(\phi\)-pseudo symmetric \(\alpha\)-para Kenmotsu manifold, the Ricci tensor \(S\).
is of the form
\[ S(Y, W) = -\alpha^2 (g(Y, W) - \eta(Y, \xi) - u(\xi)) \] (5.11)

In particular, if \( u = 0 \), then from (5.11) we get \( S(Y, W) = -\alpha^2 g(Y, W) \).

This leads to the following:

Corollary 5.4. A \( \varphi \)-symmetric \( \alpha \)-para Kenmotsu manifold is an Einstein manifold.

5. \( \Phi \)-Pseudo Ricci Symmetric \( \alpha \)-Para Kenmotsu Manifolds

Definition 6.2. [10] A \( \alpha \)-Para Kenmotsu manifold \( (\mathcal{M}^\alpha, g) \) is said to be \( \varphi \)-pseudo Ricci symmetric if the Ricci operator \( Q \) satisfies

\[ \varphi^2 (\nabla \varphi Q)(Y) = 2u(X)QY + u(Y)QX + S(Y, X) \rho \] (6.1)

for any vector field \( X, Y \), where \( u \) is a non-zero 1-form.

In particular, if \( u = 0 \), then (6.1) turns into the notion of \( \varphi \)-Ricci symmetric \( \alpha \)-Para Kenmotsu manifold introduced by Shukla and Shukla [28].

Let us take a \( \alpha \)-Para Kenmotsu manifold \( (\mathcal{M}^\alpha, g) \), which is \( \varphi \)-pseudo Ricci symmetric. Then by virtue of (2.1) it follows, from (6.1),

\[ (\nabla \varphi Q)(Y) = -\eta(\nabla \varphi Q)(Y) - \eta(X)Q + S(Y, X) \rho, \] (6.2)

from which we get

\[ g(\nabla \varphi Q)(Y), Z) = \nabla(Y, Z) + u(Y)S(X, Z) + S(Y, X)u(Z). \]

Taking \( Y = \xi \) in (6.3) and using (2.10) and (2.14), we have

\[ \alpha - u(\xi)S(X, Z) = -2g(\alpha(X, -\xi)\xi), Z) + 2[\eta(\alpha(X, -\xi)\xi)] - 2\alpha^2 \eta(X)\eta(\xi) + 4a^2 \eta(Z)u(X) - 2a^2 \eta(X)u(Z). \]

so we state (6.3) (6.4)

Theorem 6.5. In a \( \varphi \)-pseudo Ricci symmetric \( \alpha \)-para Kenmotsu manifold, the Ricci tensor is of the form (6.4).

In particular, if \( u = 0 \) then from (6.4), we get

\[ S(X, Z) = -2g(X, Z) + 2(1 - \varphi^2) \eta(X)\eta(Z), \] which implies that

the manifold under consideration is \( \eta \)-Einstein.

6. Semi-Generalized \( \Phi \)-Recurrent \( \alpha \)-Para Kenmotsu Manifolds

Definition 7.3. A \( \alpha \)-Para Kenmotsu manifold \( (\mathcal{M}^\alpha, g) \) is called semi-generalized \( \varphi \)-recurrent if its curvature tensor \( R \) satisfies the condition

\[ \varphi^2 (\nabla R)(X, Y, Z) = u(W)R(X, Y, Z) + v(W)g(Y, Z)X, \] (7.1)

for all \( X, Y, Z, W \in T\mathcal{M} \), where \( u \) and \( v \) are two 1-forms, \( v \) is non-zero and these are defined by

\[ u(W) = g(W, p_1), v(W) = g(W, p_2), \] (7.2)

where \( p_1 \) and \( p_2 \) are the vector fields associated to the 1-forms \( u \) and \( v \) respectively.

Let us consider a semi-generalized \( \varphi \)-recurrent \( \alpha \)-para Kenmotsu manifold. Then by virtue of (2.1) and (7.1) we have

\[ g(\nabla R)(X, Y, Z) = u(W)R(X, Y, Z) + v(W)g(Y, Z)X, \] (7.3)

which implies that

\[ g(\nabla R)(X, Y, Z) = v(W)g(Y, Z)X, \] (7.4)

Let \( \{e_i\} \), \( i = 1, 2, \ldots, n \) be an orthonormal basis of the tangent space at any point of the manifold. Then setting \( X = U = e_i \) in (7.4) and taking summation over \( i = 1, 2, \ldots \) we get

\[ g(\nabla R)(Y, Z) = \sum g(\nabla R)(e_i, Y, Z) = g(\nabla R)(e_i, Y, Z) = g(\nabla R)(e_i, Y, Z), \] (7.5)

The second term of (7.5) by putting \( Z = \xi \) take the form \( g(\nabla R)(e_i, Y, e_i) \).

Consider (7.5)

\[ g(\nabla R)(e_i, Y, e_i) = g(\nabla R)(e_i, Y, e_i) = g(\nabla R)(e_i, Y, e_i) = g(\nabla R)(e_i, Y, e_i) = g(\nabla R)(e_i, Y, e_i), \] at \( p \in M \). Since \( \{e_i\} \) is an orthonormal basis, so \( \nabla \varphi e_i = 0 \) at \( p \). Using (2.1) (2.4) and (2.11) in (7.6) we obtain (7.6)

\[ g(\nabla R)(e_i, Y, e_i) = g(\nabla R)(e_i, Y, e_i) = g(\nabla R)(e_i, Y, e_i) = g(\nabla R)(e_i, Y, e_i). \] (7.7)

and

\[ g(\nabla R)(e_i, Y, e_i) = g(\nabla R)(e_i, Y, e_i) = g(\nabla R)(e_i, Y, e_i) = g(\nabla R)(e_i, Y, e_i). \] (7.8)
Since $\nabla W g = 0$, we get
\[
g(\nabla W R(e_1, Y)\xi, \xi) = -g(R(e_1, Y)\xi, \nabla W \xi) = 0, \quad (7.9)
\]
which implies
\[
g(\nabla W R(e_1, Y)\xi, \xi) = -g(R(e_1, Y)\xi, \nabla W \xi) = 0, \quad (7.10)
\]
Using (2.14) in (7.10), we get
\[
g(\nabla W R(e_1, Y)\xi, \xi) = 0. \quad (7.11)
\]
Replacing $Z$ by $\xi$ in (7.5) and using (2.4), (2.6), (2.10) and (7.11), we have
\[
(\nabla W S)(Y, \xi) = 0. \quad (7.12)
\]
By using (2.14) and (2.10) in the relation
\[
(\nabla W S)(Y, \xi) = -\nabla W S(Y, \xi) - S(\nabla W Y, \xi) - S(Y, \nabla W \xi),
\]
\[
\text{it follows that}
\]
\[
(\nabla W S)(Y, \xi) = 0. \quad (7.13)
\]
In view of (7.12) and (7.13), we have
\[
-2u^2g(W, Y) - uS(Y, W) = 0. \quad (7.14)
\]
Put $Y = \xi$ in (7.14). Then by using (2.1), (2.4) and (2.10), we get
\[
-2u^2g(W, Y) + 3v(W) = 0, \quad (7.15)
\]
which reduces (7.14) to the form
\[
S(Y, W) = -2u^2g(Y, W). \quad (7.16)
\]
This leads to the following:

Theorem 7.6. A semi-generalized $\varphi$-recurrent $\alpha$-para Kenmotsu manifold $(M^{n}, g)$ is an Einstein manifold and more over the 1-forms $u$ and $v$ are related as
\[
-2u^2g(W, Y) + 3v(W) = 0
\]

7. Semi Generalized Projectively $\Phi$-Recurrent $\alpha$-Para Kenmotsu Manifold

Definition 8.4.: A $\alpha$-para Kenmotsu manifold is said to be projectively $\varphi$-recurrent manifold if there exists a non zero 1-form $u$ such that
\[
\varphi^2(\nabla W P)(X, Y)Z = u(W)P(X, Y)Z. \quad (8.1)
\]
Definition 8.5.: A $\alpha$-para Kenmotsu manifold is called semi-generalized projectively $\varphi$-recurrent if its projectively curvature tensor $P$ defined in (1.1) satisfies the condition
\[
\varphi^2(\nabla W P)(X, Y, Z) = u(W)P(X, Y)Z + v(W)g(Y, Z)X. \quad (8.2)
\]

References


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