Dominated coloring of mycielskian graphs

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Abstract

A dominated coloring is a proper coloring of a graph $G$ in such a way that each color class is dominated by a vertex of $G$. The minimum cardinality among all dominated colorings of a graph $G$ is called the dominated chromatic number of $G$, denoted by $\chi_{dom}(G)$. In this paper we give exact values for dominated coloring of mycielskian graphs and iterated mycielskian graphs. Also we give bounds for generalized mycielskian graphs.

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1 Introduction

Let $G = (V, E)$ be a finite, undirected graphs with neither loops nor multiple edges. The open neighborhood of a vertex $v$ is the set $N(v) = \{u : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$ is defined to be the closed neighborhood $N[v]$ of $v$. For a subset $S \subseteq V$, we define $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$. For a nonempty subset
$S \subseteq V$ of a graph $G$, the induced subgraph $\langle S \rangle$ is the maximal subgraph of a graph $G$ with vertex set $S$. Thus two vertices in $\langle S \rangle$ are adjacent if and only if they are adjacent in $G$.

A proper vertex coloring is a coloring to the vertices of $G$ in such a way that no two adjacent vertices receive same color. The chromatic number of a graph $G$, denoted by $\chi(G)$, is the minimum number of colors required for a proper coloring of $G$. Let $c_1, c_2, \ldots, c_k$ be the color classes of a graph $G$. If each color class is dominated by at least one vertex, then such a coloring is called a dominated coloring of a graph $G$. The minimum cardinality among all dominated colorings of a graph $G$ is called the dominated chromatic number of $G$, denoted by $\chi_{dom}(G)$. The idea of dominated coloring was presented by Merouane et al. [8]. It was shown by Merouane et al. [8] that dominated coloring problem, for any fixed $k \geq 4$, is NP-complete for arbitrary graphs. In this paper we give exact values for dominated coloring of mycielski graphs. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_k\}$ be a dominated coloring of a graph $G$. Suppose $|\mathcal{C}| = \chi_{dom}(G)$, then we say $\mathcal{C} = \{c_1, c_2, \ldots, c_k\}$ to be a $\chi_{dom}$-coloring of $G$. A vertex $v \in V(G)$ dominates a color class $c_i \in \mathcal{C}$ if $N[v] \supseteq c_i$. In this case we also say that a color class $c_i$ is dominated by a vertex $v \in V(G)$.

In search for triangle-free graphs with arbitrarily large chromatic number, Mycielski [9] developed an interesting graph transformation which transforms a graph $G$ into a graph $M(G)$, called as Mycielskian of $G$. Let $G$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The Mycielskian of a graph $G$ is the graph $M(G)$ with vertex set $V \cup W \cup \{u\} = \{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n, u\}$ and the edge set $E(M(G)) = E(G) \cup \{v_i w_j : v_i v_j \in E\} \cup \{w_i u : w_j \in W\}$. The vertex $w_i$ is called the twin vertex of $v_i$ and the vertex $u$ is the root vertex of $M(G)$. For recent results on Mycielskian of a graph $G$ we refer to [1, 4, 5, 10, 11].

### 2 Mycielskian graph

In this section, we show that if $G$ has a dominated coloring of cardinality $\chi_{dom}(G)$, then mycielski construction produces a $\chi_{dom}(G)+1$-coloring of $M(G)$. Further we show that $\chi_{dom}(M^k(G)) = \chi_{dom}(G)+k$, where $M^k(G)$ is the iterative mycielskian of $G$. 

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Theorem 1. For a graph $G$ we have $\chi_{\text{dom}}(M(G)) = \chi_{\text{dom}}(G) + 1$.

Proof. Let $C = \{c_1, c_2, \ldots, c_k\}$ be a $\chi_{\text{dom}}$-coloring of $G$. Now consider a coloring $C_1 = C \cup \{u\}$ of $M(G)$, where each twin vertex $v'$ receives the color of $v$ and the root vertex $u$ is assigned a new color. Let $c_1 = \{u\} \in C_1$. Clearly each color class $c_i, i \neq 1$, is dominated by some vertex $v \in V$ and the color class $c_1 = \{u\}$ dominates itself. Thus $\chi_{\text{dom}}(M(G)) \leq \chi_{\text{dom}}(G) + 1$.

Next we claim that $\chi_{\text{dom}}(M(G)) \geq \chi_{\text{dom}}(G) + 1$. Let $C_1 = \{c_1, c_2, \ldots, c_k\}$ be a $\chi_{\text{dom}}$-coloring of $M(G)$. We now claim that there exists a $\chi_{\text{dom}}$-coloring of $G$ using $k - 1$ colors.

**case i.** $c_1 = \{u\} \in C_1$ of $M(G)$.

Now consider a restricted coloring $C_2$ of $C_1$ to $G$ as follows. If $c_i \in C_1$ such that $c_i \subset V'$, then choose a representative vertex $\alpha' \in c_i$ in $V'$ and recolor its twin vertex $\alpha \in V$ by color $i$. Clearly the above restricted coloring is a proper coloring of $G$. We now show that each color class $c_i, i \neq 1$, is dominated by some vertex $v \in V(G)$. If a vertex $v \in V(G)$ dominates a color class $c_i \in C_1$ such that $c_j \cap V(G) \neq \phi$, then $v$ still continues to dominate the color class $c_j$ in the restricted coloring $C_2$ of $G$. Suppose $v \in V(G)$ dominates a color class $c_i \in C_1$ such that $c_i \subset V'$. Then it takes cue from the definition of $M(G)$ that the vertex $v$ dominates the color class $c_i$ in the restricted coloring $C_2$ of $G$, since the twin vertex of the representative vertex $\alpha' \in c_i$ receives the color $i$. Hence $C_2$ is a dominated coloring of $M(G)$ using $k - 1$ colors. Thus $\chi_{\text{dom}}(G) \leq \chi_{\text{dom}}(M(G)) - 1$ and hence equality holds.

**case ii.** $c_1 \neq \{u\} \in C_1$, that is $S = c_1 \cap V(G) \neq \phi$.

Now consider a restricted coloring $C_3$ of $C_1$ to $G$ as follows. We recolor each vertex of $S$ by the color of its twin vertex. If $c_i \in C_1$ such that $c_i \subset V'$, then choose a representative vertex $\alpha' \in c_i$ in $V'$ and recolor its twin vertex $\alpha \in V$ by color $i$. Then it follows from case i, that $\chi_{\text{dom}}(G) \leq \chi_{\text{dom}}(M(G)) - 1$ and hence equality holds. \qed

**Definition 2.** The iterative mycielskian of a graph $G$ denoted by $M^k(G)$, is as follows $M^k(G) = M(M^{k-1}(G))$, where $k \geq 1$ and $M^0(G) = G$.

As a consequence of Theorem 1, we obtain the following result.
Corollary 3. Let $G$ be a graph. Then $\chi_{dom}(M^k(G)) = \chi_{dom}(G) + k$.

3 Generalized Mycielskian Graph

In this section, we give general bounds for generalized mycielskian graph $M_k(G)$ and prove that the bounds are sharp.

Let $G$ be a graph with vertex set $V^0 = \{v^0_1, v^0_2, \ldots, v^0_n\}$ and edge set $E^0$.

Given an integer $k \geq 1$, the $k$-Mycielskian of $G$, denoted by $M_k(G)$, is the graph $M_k(G)$ with vertex set $V^0 \cup V^1 \cup V^2 \cup \cdots \cup V^k \cup \{u\}$, where $V^i = \{v^i_j : v^0_j \in V^0\}$ is the $i$th distinct copy of $V^0$ for $i = 1, 2, \ldots, k$ and edge set $E^0 \cup (\bigcup_{i=0}^{k-1}\{v^j_i v^{i+1}_j : v^0_j \in E^0\}) \cup \{v^k_j u : v^k_j \in V^k\}$.

In the following result we prove that for any $\chi_{dom}$-coloring of $M_k(G)$, the $N|V^{i+1}|$, $1 \leq i \leq k-2$, contains at least 2 color classes.

Lemma 4. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_n\}$ be a $\chi_{dom}$-coloring of $M_k(G)$. Then the set $V^i \cup V^{i+1} \cup V^{i+2}, 1 \leq i \leq k-2$, contains at least two color classes of $\mathcal{C}$ in $M_k(G)$.

Proof. Let $v^j_k \in V^j, 1 \leq k \leq n$. Suppose $\{v^j_k\} \in \mathcal{C}$, that is the vertex $v^j_k$ dominates itself. Then at least one more color class, say $c_a$, is needed to color the vertices of $V^j - \{v^j_k\}$ in $M_k(G)$. By definition of $M_k(G)$, it follows that the color class $c_a$ of $\mathcal{C}$ in $M_k(G)$ is dominated by one of the vertices $v^{j+1}_k$ or $v^{j+2}_k$. Suppose $\{v^j_k\} \notin \mathcal{C}$.

Let $v^j_k \in c_a$. Since $\mathcal{C}$ is a dominated coloring of $M_k(G)$, it follows that there exists a vertex $v^{j+1}_l, l \neq k$, which dominates the color class $c_a$ in $M_k(G)$. It is clear that $c_a \neq V^j \cup V^{j+2}$. Then there exists a vertex, say $v^j_l, l \neq k$, such that $v^j_l$ receives the color $c_b, b \neq a$, of $\mathcal{C}$ in $M_k(G)$. Since $\mathcal{C}$ is a dominated coloring of $M_k(G)$, it follows that there exists a vertex, say $v^{j+1}_k, k \neq l$, which dominates the color class $c_b$ of $\mathcal{C}$ in $M_k(G)$. Clearly $c_a \cup c_b \subseteq \{V^i \cup V^{i+1} \cup V^{i+2}\}, 1 \leq i \leq k-2$. Hence the proof.

Remark 5. Let $A_{i,i+3} = \{V^i \cup V^{i+1} \cup V^{i+2} \cup V^{i+3}\}, 1 \leq i \leq k-3$, be a set of $M_k(G)$. Then it takes cue from Lemma 4, that each set $A_{i,i+3}, 1 \leq i \leq k-3$, has at least four color classes in any $\chi_{dom}$-coloring of $M_k(G)$.
Theorem 6. Let \( G \) be a connected graph. Then

\[
\chi_{\text{dom}}(G) + k \leq \chi_{\text{dom}}(M_k(G)) \leq \ \begin{cases} 
\left\lceil \frac{k}{2} \right\rceil n + 1, & \text{if } k \text{ is odd} \\
\left\lceil \frac{k}{2} \right\rceil n + 2, & \text{if } k \text{ is even}
\end{cases}
\]

Proof. Let \( \mathcal{C} = \{c_1, c_2, \ldots, c_{\chi_{\text{dom}}(G)}\} \) be a \( \chi_{\text{dom}} \)-coloring of \( M_k(G) \), where every vertex of \( c_i \) receives the color \( i \). Now we claim that \( \chi_{\text{dom}}(M_k(G)) \geq \chi_{\text{dom}}(G) + k \).

Case i. \( k = 4r \).

Since \( (V^0) = G \), we need at least \( \chi_{\text{dom}}(G) \) colors to color the vertices of \( V^0 \cup V^1 \). Then the \( r \) sets \( A_{i+3}, i = 2, 6, 10, \ldots, 4r - 6 \) are vertex disjoint in \( M_k(G) \). By remark 5, the sets \( A_{i+3}, i = 2, 6, 10, \ldots, 4r - 6 \), contain at least \( 4r - 4 \) colors of \( \mathcal{C} \), say \( 1, 2, \ldots, 4r - 4 \), in any \( \chi_{\text{dom}} \)-coloring of \( M_k(G) \). By lemma 4, we need at least two colors to color the vertices of \( N[V^{4r-1}] \) and one more color class to color the root vertex \( u \). Thus

\[
\chi_{\text{dom}}(M_k(G)) \geq \chi_{\text{dom}}(G) + 4r - 4 + 3 = \chi_{\text{dom}}(G) + 4r - 1. \]

Let \( \mathcal{C}_1 = \{c_1, c_2, \ldots, c_{\chi_{\text{dom}}(G)+4r-1}\} \) be a \( \chi_{\text{dom}} \)-coloring of \( M_k(G) \). Clearly \( \chi_{\text{dom}}(G) \) colors are contained in \( V^0 \cup V^1 \). By lemma 4, we may assume that \( c_{i-1}, c_i \subseteq N[V^{i+1}], 2 \leq i \leq 4r - 1 \). Since \( \chi_{\text{dom}}(M(G)) = \chi_{\text{dom}}(G) + 4r - 1 \) and \( c_{i-1}, c_i, c_{i+1}, c_{i+2} \subseteq A_{i+3}, 2 \leq i \leq 4r - 6 \), it follows that \( c_{\chi_{\text{dom}}(G)+4r-3}, c_{\chi_{\text{dom}}(G)+4r-2} \subseteq N[V^{4r-1}] \) and \( \{u\} \subseteq c_{\chi_{\text{dom}}(G)+4r-1} \). Then the color class \( c_{\chi_{\text{dom}}(G)+4r-3} = V^{4r-2} \cup V^r \) and \( c_{\chi_{\text{dom}}(G)+4r-2} = V^{4r-1} \). Clearly the color classes \( c_{\chi_{\text{dom}}(G)+4r-3} \) and \( c_{\chi_{\text{dom}}(G)+4r-2} \) are not dominated by any vertex of \( M(G) \). Hence

\[
\chi_{\text{dom}}(M_k(G)) \geq \chi_{\text{dom}}(G) + 4r = \chi_{\text{dom}}(G) + k.
\]

Case ii. \( k = 4r + 1 \).

Since \( (V^0) = G \), we need at least \( \chi_{\text{dom}}(G) \) colors to color the vertices of \( V^0 \cup V^1 \). Then the \( r \) sets \( A_{i+3}, i = 2, 6, 10, \ldots, 4r - 6 \), contain at least \( 4r - 2 \) colors of \( \mathcal{C} \), say \( 1, 2, \ldots, 4r - 2 \), in any \( \chi_{\text{dom}} \)-coloring of \( M_k(G) \). Further we need one more color class, say \( c_{\alpha}, \alpha \notin \{1, 2, \ldots, \chi_{\text{dom}}(G) + 4r\} \), to color the root vertex \( u \). Thus \( \chi_{\text{dom}}(M_k(G)) \geq \chi_{\text{dom}}(G) + 4r + 1 = \chi_{\text{dom}}(G) + k \).

Case iii. \( k = 4r + 2 \).

Since \( (V^0) = G \), we need at least \( \chi_{\text{dom}}(G) \) colors to color the vertices of \( V^0 \cup V^1 \). Then the \( r \) sets \( A_{i+3}, i = 2, 6, 10, \ldots, 4r - 6 \), contain at least \( 4r - 2 \) colors of \( \mathcal{C} \), say \( 1, 2, \ldots, 4r \), in any \( \chi_{\text{dom}} \)-coloring of \( M_k(G) \). Further to color the
vertices of $V^{4r+2} \cup \{u\}$, we need at least two more color classes, say $c_\alpha$ and $c_\beta$, where $\{\alpha, \beta\} \notin \{1, 2, \ldots, \chi_{dom}(G) + 4r\}$. Thus $
chi_{dom}(M_k(G)) \geq \chi_{dom}(G) + 4r + 2 = \chi_{dom}(G) + k$.

**Case iv.** $k = 4r + 3$.

Since $(V^0) = G$, we need at least $\chi_{dom}(G)$ colors to color the vertices of $V^0 \cup V^1$. Then the $r$ sets $A_{i,t+3}, i = 2, 6, 10, \ldots, 4r - 6, 4r - 2$ are vertex disjoint in $M_k(G)$. By remark 5, the sets $A_{i,t+3}, i = 2, 6, 10, \ldots, 4r - 6, 4r - 2$, contains at least $4r$ colors of $C$, say $1, 2, \ldots, 4r$, in any $\chi_{dom}$-coloring of $M_k(G)$. Further to color the vertices of $V^{4r+2} \cup V^{4r+3} \cup \{u\}$, we need at least two more color classes, say $c_\alpha$ and $c_\beta$, where $\{\alpha, \beta\} \notin \{1, 2, \ldots, \chi_{dom}(G) + 4r\}$. Thus $\chi_{dom}(M_k(G)) \geq \chi_{dom}(G) + 4r + 2$. Suppose $\chi_{dom}(M_k(G)) = \chi_{dom}(G) + 4r + 2$. Let $C_1 = \{c_1, c_2, \ldots, c_{\chi_{dom}(G) + 4r+2}\}$ be a $\chi_{dom}$-coloring of $M_k(G)$. Clearly $\chi_{dom}(G)$ colors are contained in $V^0 \cup V^1$. By lemma 4, we may assume that $c_{i-1}, c_i \subseteq N[V^{i+1}], 2 \leq i \leq 4r - 1$.

Since $\chi_{dom}(M(G)) = \chi_{dom}(G) + 4r + 2$ and $c_{i-1}, c_i, c_{i+1}, c_{i+2} \subseteq A_{i,t+3}, 2 \leq i \leq 4r - 6$, it follows that $c_{\chi_{dom}(G) + 4r+2}, c_{\chi_{dom}(G) + 4r+3} \subseteq V^{4r+2} \cup V^{4r+3} \cup \{u\}$. Then the color class $c_{\chi_{dom}(G) + 4r+2} = V^{4r+2} \cup \{u\}$ and $c_{\chi_{dom}(G) + 4r+3} = V^{4r+3}$. Clearly the color classes $c_{\chi_{dom}(G) + 4r+2}$ is not dominated by any vertex of $M(G)$. Thus $\chi_{dom}(M_k(G)) \geq \chi_{dom}(G) + 4r + 3 = \chi_{dom}(G) + k$.

Now we prove that 'less than or equal to' inequality of the theorem holds.

Suppose $k = 4r$.

Let $C = \{v^i, v^{i+2}_i : 1 \leq i \leq n \text{ and } j \equiv 0(\text{mod } 4) \text{ or } j \equiv 1(\text{mod } 4)\}$.

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be a coloring of $M_k(G)$. Clearly $C$ is a proper coloring of $M_k(G)$.

Suppose $k = 4r$. Then the color class $\{v^i, v^{i+2}_i, 1 \leq i \leq n\}$ is dominated by a vertex of $V^{j+1}, if j \equiv 0(\text{mod } 4)$ or by a vertex of $V^{j+2}, if j \equiv 1(\text{mod } 4)$. Further the color class $\{u\}$ is dominated by
itself. Suppose \( k = 4r + 1 \). Then the color class \( \{v_j^i, v_j^{i+2}\}, 1 \leq i \leq n \), is dominated by a vertex of \( V^{j+1} \), if \( j \equiv 0(\text{mod} \ 4) \) or by a vertex of \( V^{j+2} \), if \( j \equiv 1(\text{mod} \ 4) \). Further the root vertex \( u \) dominates the color classes \( V^k \) and \( \{u\} \). Suppose \( k = 4r + 2 \). Then the color class \( \{v_j^i, v_j^{i+2}\}, 1 \leq i \leq n \), is dominated by a vertex of \( V^{j+1} \), if \( j \equiv 0(\text{mod} \ 4) \) or by a vertex of \( V^{j+2} \), if \( j \equiv 1(\text{mod} \ 4) \). Further the root vertex \( u \) dominates the color class \( V^k \) and \( \{u\} \). Finally the root vertex \( u \) dominates the color class \( V^k \). Suppose \( k = 4r + 3 \). Then the color class \( \{v_0^i, v_1^i\}, 1 \leq i \leq n \), is dominated by a vertex of \( V^0 \). Further the color class \( \{v_j^i, v_j^{i+2}\}, 1 \leq i \leq n \), is dominated by a vertex of \( V^{j+1} \), if \( j \equiv 0(\text{mod} \ 4) \) or by a vertex of \( V^{j+2} \), if \( j \equiv 1(\text{mod} \ 4) \). Finally the root vertex \( u \) dominates the color classes \( V^k \) and \( \{u\} \). Hence 'less than or equal to' inequality of the theorem holds. 

The above bounds are sharp. For the graph \( G = K_n \) we have \( \chi_{\text{dom}}(M(G)) = \chi_{\text{dom}}(G) + k \) and for the graph \( G = H \circ K_1 \), where \( H \) is any connected graph, the upper bound is satisfied.

References


