An Inventory System with replacement and postponed demands

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Abstract

In this paper we consider a continuous review perishable inventory system with a finite number of homogeneous sources of demands. The arrival of a demand occurs according to a quasi-random process. The life time of an item is exponentially distributed. The inventory is replenished according to \((s, S)\) policy and the perished items are replaced by the supplier at the time of replenishment of an order with free of cost. The lead time is assumed to be exponential. When the on-hand inventory level is zero, any newly arriving demand offered for a choice of postponement according to a Bernoulli trial. The selection of postponed demands are made according to some prefixed rule. The time between any two successive selections is assumed to be distributed as exponential. The joint probability distribution of the on-hand inventory level, the number of perished items which are stored for replacement and the number of customers in the pool is obtained in the steady state case. Various measures of system performance in the steady state case are determined. We numerically determined the optimal values for minimizing the cost function.
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1 Introduction

Inventory system with postponed demands has received attention in the last few decades. Any arriving demands who find the on-hand inventory level is zero are either lost or backlogged. In the backlogging case, the backlogged demands are fully or partially delivered, immediately after the replenishment of an ordered item. But in some real-life situations, the backlogged demand may have to wait in a place, called pool even after replenishment. This type of inventory model is symbolized as Inventory System with Postponed demands. In the literature, inventory system with postponed demands concept was introduced by Berman et. al. [1]. Krishnamoorthy and Islam [2] dealt an \((s, S)\) Inventory system with postponed demands in which the arriving demands who finds the inventory level is zero join the pool and these pooled customers are selected according to an exponential distributed time.

Traditional inventory models are no longer applicable because these inventory models assume that all the inventories in the system are used to fulfill the future demands. The inventory systems with perishable products has a special significance as it influences the cost and wage of a retailer. For instance, pharmaceutical items could be used before the expiry date, otherwise it will become valueless. Similarly, in the paper of Sivakumar and Arivarignan [9], they considered a perishable inventory in which the demands occur during the stock-out period enters into the pool according to Bernoulli trial and the inter-selection time of pooled customers are assumed to be an exponential distribution. Paul Manuel et al. [3] extended the above model with negative demands, who will remove the waiting demand from the tail of the pool. For review articles on perishable inventory systems, see Nahmias [4, 5].

The concept of finite population has been motivated by the fact that the number of potential demands is far less than infinite. In this concept, the arriving demand may depend on the number of demands already in the system. This process is denoted as quasi-

In all the above models the perished/failed items are considered to be lost. In any inventory models, the decay, spoilage or expiry of an items are really harmful and worthless and also causes significant profit loss to such models. For these inventory models, replacement policies act as safeguards to the retailers dealing with the perishable/failed items. Sivakumar and Anbazhagan[8] studied a continuous review inventory system with replaceable items in which they considered the base stock policy.

In this work, we combine the concept of finite source and the inventory system with replaceable items and derive the steady state probability vector. The rest of the paper is organized as follows. In the section 2, we describe the mathematical model intuitive with the problem. Section 3, proceed by the study of stability and steady-state analysis of the model. In section 4, we derive various system performance measures in the steady state and calculate the long-run total expected cost rate. In the final section we present the numerical illustrations from the results.

2 Problem Formulation

Consider a continuous review perishable inventory system in which the demands are generated by a finite number (N) of homogeneous sources and the occurrence of demand forms a quasi-random process with parameter \( \lambda \). That is, the probability that any particular source generates a demand in any interval \((t, t + dt)\) is \( \lambda dt + o(dt) \) where \( \frac{o(dt)}{dt} \to 0 \) as \( dt \to 0 \) if the source is idle at time \( t \), and zero otherwise. The life time of an item in the inventory is exponentially distributed with parameter \( \gamma \). The demand occurs during the stock-out period, is offered a choice of either leaving the system immediately or being postponed. The demands those who accept the postponement joins a pool. We assume that the demands join the pool according to independent Bernoulli trails with probability \( p \) and the remaining leave the system. The pooled demands are selected if the on-hand inventory level is above the reorder level.
or the on-hand inventory is greater than the arriving demands (i.e., \( L(t) > N - X(t) \)), otherwise only the arriving demand will be satisfied. The inter-selection time between two successive selections is assumed to have an exponential distribution with parameter \( \theta (> 0) \). The inventory is replenished according to \((s, S)\) ordering policy and the ordering quantity is adjusted according to the following manner. When the inventory level reaches the reorder level \( s \), an order for \( Q = (S - i) - k \) items are placed along with \( k, k = 0, 1, 2, \ldots, S - s \) perishable items are replaced. At the time of receiving the order, the perished items will be replaced without any cost. The lead time is exponentially distributed with parameter \( \beta (> 0) \).

3 Analysis

Let \( L(t), F(t), X(t) \) respectively, denote the on-hand inventory level, the number of perished items, the number of demands in the pool at time \( t \). From our assumptions it can be shown that the triplet \( \{L(t), F(t), X(t), t \geq 0\} \) is a continuous time Markov chain with state space

\[
\Omega = \{(i, f, x) : i \in E_s, f \in E_{S-i}, x \in E_N\} \cup \{(i, f, x) : i \in E_{S+1}, f \in E_{S-i}, x \in E_N\}
\]

To induce an order on the state space we define the following ordered tuplets

\[
\langle<i>\rangle = \begin{cases} 
\langle<i,0>,<i,1>,\ldots,<i,s-i>\rangle & i = 0, 1, \ldots, s; \\
\langle<i,0>,<i,1>,\ldots,<i,S-i>\rangle & i = s+1, s+2, \ldots, S;
\end{cases}
\]

\[
\langle<i,f,x>\rangle = \begin{cases} 
\langle<i,f,x>\rangle & i = 0, 1, 2, \ldots, s; f = 0, 1, 2, \ldots, i; x = 0, 1, 2, \ldots, N \\
\langle<i,f,x>\rangle & i = s+1, s+2, \ldots, S; f = 0, 1, 2, \ldots, S+1; x = 0, 1, 2, \ldots, N
\end{cases}
\]

Hence, the state space \( E \) is ordered as

\( \langle<i>\rangle, \langle<i,1>\rangle, \ldots, \langle<i,S>\rangle \).

The infinitesimal generator \( P \) of this process can be written in a block-partitioned matrix as follows

\[
[P]_{ij} = \begin{cases} 
A_i, & j = i, i = 0, 1, \ldots, S \\
B_i, & j = i + 1, i = 0, 1, 2, \ldots, S \\
C_i, & j = i + Q, i = 0, 1, \ldots, s \\
0 & \text{otherwise}
\end{cases}
\]

where

\[
A_i = I_{s+1} + \beta N_{i+1}
\]

\[
B_i = I_{s+1} + \tilde{A}
\]

\[
C_i = I_{s+1} + \beta N_{i+1}
\]

\[
A_0 = I_{s+1} + \tilde{A}
\]
\[ [A]_{kl} = \begin{cases} 
-((\beta + p(N - k)\lambda) l - k) & l = k \quad k = 0, 1, \ldots, N \\
p(N - k)\lambda l - k + 1 & l = k + 1 \quad k = 0, 1, \ldots, N - 1 \\
0 & \text{otherwise}
\end{cases} \]

for \( i = 1, 2, \ldots, s \)

\[ A_i = -I_{s-i+1} \otimes \mathbf{1}, \]

\[ [A]_{kl} = \begin{cases} 
-(N-k)\lambda + \gamma \gamma & l = k \quad k = 0, 1, \ldots, N-1 \\
n(N-k)\lambda + \gamma \gamma + \theta_k & l = k + 1 \quad k = N - i + 1, N - i + 2, \ldots, N \\
0 & \text{otherwise}
\end{cases} \]

for \( i = s + 1, s + 2, \ldots, S \)

\[ A_i = -I_{S-i+1} \otimes [(N-k)\lambda + \theta_k + \gamma I) \mathbf{1} \otimes \mathbf{1}] \]

for \( i = 1, 2, \ldots, s \)

\[ [B]_{kl} = \begin{cases} 
1 & l = k, k = 0, 1, \ldots, s-i \\
0 & l = k + 1, k = 0, 1, \ldots, s-i \\
0 & \text{otherwise}
\end{cases} \]

\[ [\theta]_{kl} = \begin{cases} 
1 & l = k, \quad k = 0, 1, \ldots, S-i \\
0 & l = k + 1, \quad k = 0, 1, \ldots, S-i \\
\theta_k & l = k \quad k = 0, 1, \ldots, S-i \\
0 & \text{otherwise}
\end{cases} \]

\[ [\beta]_{kl} = \begin{cases} 
1 & l = k, \quad k = 0, 1, \ldots, S-i \\
0 & l = k + 1, \quad k = 0, 1, \ldots, S-i \\
0 & \text{otherwise}
\end{cases} \]

\[ [\beta]_{kl} = \begin{cases} 
1 & l = k, \quad k = 0, 1, \ldots, S-i \\
0 & l = k + 1, \quad k = 0, 1, \ldots, S-i \\
0 & \text{otherwise}
\end{cases} \]

\[ [\beta]_{kl} = \begin{cases} 
1 & l = k, \quad k = 0, 1, \ldots, S-i \\
0 & l = k + 1, \quad k = 0, 1, \ldots, S-i \\
0 & \text{otherwise}
\end{cases} \]

\[ \mathbf{Q} = \begin{pmatrix} 
\theta_1 & \theta_2 & \cdots & \theta_s \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix} \]

\[ \mathbf{Q}(N+1) \times (N+1) \]

\[ \mathbf{Q} = \begin{pmatrix} 
\theta_1 & \theta_2 & \cdots & \theta_s \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix} \]

\[ \mathbf{Q}(N+1) \times (N+1) \]

3.1 Steady state analysis

It can be seen from the structure of \( P \) that the homogeneous Markov Process \( \{ (L(t), F(t), X(t)) t \geq 0 \} \) on the finite state space \( \Omega \) is irreducible. Hence the limiting distribution

\[ \pi^{<i,f,x>} = \lim_{t \to \infty} \Pr[L(t) = i, F(t) = f, X(t) = x | L(0), F(0), X(0)] \]

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exists. Let $\Pi = (\pi^{<0>}, \pi^{<1>}, \ldots, \pi^{<S>})$, where

$$
\pi^{<i>} = (\pi^{<i,0>, \pi^{<i,1>, \ldots, \pi^{<i,s-i>}}}) \text{ for } i = 0, 1, \ldots, s
$$

$$
\pi^{<i>} = (\pi^{<i,0>, \pi^{<i,1>, \ldots, \pi^{<i,s-i>}}}) \text{ for } i = s+1, s+2, \ldots, S.
$$

Then the vector of limiting probabilities $\Pi$ satisfies

$$
\Pi P = 0 \text{ and } I \Pi e = 1. \quad (1)
$$

The first equation of the above yields the following set of equations:

$$
\pi^{<i>} A_i + \pi^{<i+1>} B_{i+1} = 0, \quad i = 0, 1, \ldots, Q - 1. \quad (2)
$$

$$
\pi^{<i>} A_i + \pi^{<i+1>} B_{i+1} + \pi^{<i>} C_{i+1} = 0, \quad i = Q. \quad (3)
$$

$$
\pi^{<i>} A_i + \pi^{<i+1>} B_{i+1} + \pi^{<i>} C_{i+1} = 0, \quad i = Q + 1, \ldots, S - 1. \quad (4)
$$

$$
\pi^{<i>} A_i + \pi^{<i>} C_{i+1} = 0, \quad i = S. \quad (5)
$$

The above equations (except (3)) can be recursively solved to get

$$
\pi^{<i>} = \pi^{<Q>} C_i, \quad i = 1, \ldots, S
$$

where

$$
\alpha_i = \left\{\begin{array}{ll}
1, & i = Q, \\
(-1)^{Q-i+1} \{(B_{Q_i} A_{Q_i-1}) \ldots (B_{i+1} A_{i+1})\}, & i = Q+1, \ldots, S-1.
\end{array}\right.
$$

with

$$
T_0 = A_{Q} C_Q, \quad T_i = (B_i A_{i-1} \ldots B_{i-1} A_{i-1} C_{i-1} + T_{i-1} B_{i-1} A_{i-1}) A_{i-1}, \quad i = 1, 2, \ldots, s-1
$$

and $\pi^{<Q>}$ can be obtained by solving $\pi^{<Q>} A_Q + \pi^{<Q+1>} B_{Q+1} + \pi^{<Q>} C_0 = 0$ and

$$
\pi^{<Q>} \left\{\sum_{i=0}^{Q-1} (-1)^{Q-i}(B_{Q_i} A_{Q_i-1}) \ldots (B_{i+1} A_{i+1}) + 1
$$

$$
+ \sum_{Q+1}^{S} (-1)^{Q-i+1} \{(B_{Q_i} A_{Q_i-1}) \ldots (B_{i+1} A_{i+1}) T_{Q+1-1}\} \right\} = 1.
$$

4 System performance measures

In this section, We derive some stationary performance measures of the system. These measures are used to study the qualitative behavior of our model.
1. Expected inventory level: \( \varsigma_I = \sum_{i=1}^{\infty} \varsigma^{<i>}e \).

2. Expected reorder rate:
\[
\varsigma_R = \sum_{k=0}^{Q-1} \gamma \varsigma^{<1,k,s>}e + \sum_{k=0}^{Q-1} \sum_{x=0}^{N} \varsigma^{<s+1,k,s>}e + (N - s) \lambda \sum_{k=0}^{Q-1} \sum_{x=0}^{N} \varsigma^{<s+1,k,s>}e.
\]

3. Expected perishable rate: \( \varsigma_P = \sum_{i=1}^{\infty} \varsigma^{<i>}e \).

4. Expected number of demands in the pool:
\[
\varsigma_{PC} = \sum_{i=0}^{s} \sum_{k=0}^{N} \sum_{x=0}^{N} \varsigma^{<i,k,s>}e + \sum_{i=s+1}^{Q} \sum_{k=0}^{N} \sum_{x=0}^{N} \varsigma^{<i,k,s>}e.
\]

5. Cost Analysis: The long-run total expected cost per unit time for this system in the steady state is given by
\[
TC(s,S) = c_I \varsigma_I + c_R \varsigma_R + c_P \varsigma_P + c_{PC} \varsigma_{PC}.
\]
- \( c_I \): The inventory carrying cost per unit item per unit time.
- \( c_R \): Set-up cost per order.
- \( c_P \): Perishable cost per unit item per unit time.
- \( c_{PC} \): Waiting time cost of a customer in the pool per unit time.

Substituting the values \( \zeta_s \), we get
\[
TC(s,S) = c_I \sum_{i=1}^{\infty} \varsigma^{<i>}e +
\left( c_R \sum_{k=0}^{Q-1} \sum_{x=0}^{N} (s+1) \gamma \varsigma^{<1,k,s>}e + \sum_{k=0}^{Q-1} \sum_{x=0}^{N} \varsigma^{<s+1,k,s>}e + (N - s) \lambda \sum_{k=0}^{Q-1} \sum_{x=0}^{N} \varsigma^{<s+1,k,s>}e \right)
+ c_P \sum_{i=1}^{\infty} \varsigma^{<i>}e + c_{PC} \left[ \sum_{i=0}^{s} \sum_{k=0}^{N} \sum_{x=0}^{N} \varsigma^{<i,k,s>}e + \sum_{i=s+1}^{Q} \sum_{k=0}^{N} \sum_{x=0}^{N} \varsigma^{<i,k,s>}e \right].
\]

5 Numerical Study

A three dimensional plot of \( TC(s,S) \) is presented in Figure 1. We use simple numerical search procedure to get the optimal values of \( TC \), \( S \) and \( s \) (say \( TC^*, S^* \) and \( s^* \) respectively). The minimum value of \( TC = 16.2656 \) is obtained at \((S^*, s^*) = (26, 7)\) for fixed \( N = 10, \theta = 0.6, \beta = 0.2, \lambda = 0.9, \gamma = 0.03, p = 0.6, c_I = 0.06, c_R = 9, c_P = 0.02, c_{PC} = 4.\)
Figure 1: A three dimensional total cost rate per unit time

References


