On Some Labeling of Quadrilateral Snake

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Abstract

A quadrilateral snake $Q_n$ is obtained from a path $v_1v_2...v_n$ by joining $v_i$ and $v_{i+1}$ to two new vertices $u_j$ and $w_j$ for $1 \leq j \leq n-1$ then joining $u_j$ and $w_j$. The every edge of a path is replaced by a cycle $c_i$. In this paper the ways to construct square sum, square difference, Root Mean square, strongly Multiplicative, Even Mean, Odd Mean, Cordial and Total Cordial labeling for Quadrilateral Snake graphs are reported.

Key Words: Quadrilateral snake graph, Square sum labeling, Square difference labeling, Root Mean square labeling, Strongly Multiplicative labeling, Even Mean labeling, Odd Mean labeling, Cordial labeling and Total Cordial labeling.

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1. Introduction

The graphs considered here are finite, undirected and simple. The concept of graph labeling was introduced by A. Rosa in 1967. Let $G(V,E)$ be a $(p,q)$ graph. The Mean labeling was introduced by S.Somasundram and R.Ponraj[2]. Root square mean labeling was introduced by S.S.Sandhya, S.Somasundram and S.Anusa in [3]. The Square Sum and Square Difference labelings were introduced by Ajitha, Arumugam and Germina [4]. The Concept of strongly Multiplicative graphs was introduced by Beineke and Hegde [2001]. The Concept of Cordial, Total Cordial labeling were introduced by Sundram and Somasundram[5]. In this paper we prove that the existence of square sum, square difference, Root Mean square, Strongly Multiplicative, Even Mean, Odd Mean, Cordial and Total Cordial labelings of Quadrilateral snake $Q_n$ for $n \geq 2$.

**Definition 1.1:** A Quadrilateral Snake $Q_n$ is obtained from a path $v_1v_2...v_n$ by joining $v_i$ and $v_{i+1}$ to two new vertices $u_j$ and $w_j$ for $1 \leq j \leq n-1$ respectively and then joining $u_j$ and $w_j$. That is every edge of a path is replaced by a cycle $c_4$.

**Definition 1.3:** Let $G$ be a $(p,q)$ graph. A one-one map $f : V(G) \rightarrow \{0,\ldots, p-1\}$ is said to be a Square Sum labeling if the induced map $f^*(uv) = (f(u))^2 + (f(v))^2$ is injective. It is said to be a Square Difference labeling if the induced map $f^*(uv) = (f(u))^2 - (f(v))^2$ is injective.

**Definition 1.4:** Let $G$ be a $(p,q)$ graph. A graph $G$ admits Root Mean square labeling if there exist a injective mapping from the vertices of $G$ to set $\{0,1,2,\ldots,2p\}$ such that when each edge $uv$ is assigned the label $f^*(uv) = \sqrt{(f(u))^2 + (f(v))^2}$, then the resulting edge labels are distinct.

**Definition 1.5:** A $(p,q)$ graph said to be strongly Multiplicative if there exist a one-one map $f : V(G) \rightarrow \{1,2,\ldots,p\}$ such that the induced map defined by $f^*(uv) = f(u)f(v)$ giving distinct edge values.

**Definition 1.6:** A $(p,q)$ graph is said to be Even Mean graph if there exist a one-one map $f : V(G) \rightarrow \{2,4,\ldots,2q\}$ such that the induced map defined by $f^*(uv) = \frac{f(u) + f(v)}{2}$ giving distinct edge values.

**Definition 1.7:** A $(p,q)$ graph is called odd Mean graph if there exist a one-one map $f : V(G) \rightarrow \{1,3,5,\ldots,2q-1\}$ such that the induced map defined by $f^*(uv) = \frac{f(u) + f(v)}{2}$ giving distinct edge values.
Definition 1.8:

A mapping $f : V(G) \rightarrow \{0, 1\}$ is called binary vertex labeling of $G$ and $f(v)$ is called the label of the vertex $v$ of $G$ under $f$.

The induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$.

Definition 1.6:

A binary vertex labeling of a graph $G$ is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Where $v_f(0)$ = number of vertices of $G$ having label 0 under $f$.

$v_f(1) =$ number of vertices of $G$ having label 1 under $f$.

$e_f(0) =$ number of edges of $G$ having label 0 under $f^*$.

$e_f(1) =$ number of edges of $G$ having label 1 under $f^*$.

A graph $G$ is called cordial if it admits cordial labeling.

Definition 1.7:

A total-cordial labeling of a graph $G$ with vertex set $V$ and edge set $E$ as an cordial labeling if $|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1$.

Structural properties of $Q_n$

Throughout this paper $Q_n$ denotes a Quadrilateral snake graph on n vertices. The number of vertices in $Q_n$ is $(3n-2)$ and the number of edges is $4(n-1)$.

2. Main Results

Theorem 2.1:

The Quadrilateral Snake $Q_n$ is Square sum graph for all $n \geq 2$.

Proof: Consider a path $v_1v_2...v_n$ by joining $v_i$ and $v_{i+1}$ to two new vertices $u_j$ and $w_j$, $1 \leq j \leq n-1$.

Define a function $f : V(Q_n) \rightarrow \{0,1,...,3(n-1)\}$ by
\[ f(v_i) = i - 1, 1 \leq i \leq n \]
\[ f(u_j) = 2(n-1) + j, 1 \leq j \leq n-1 \]
\[ f(w_j) = (n-1) + j, 1 \leq j \leq n-1 \]

Thus the labels of the vertices are given below:
\[ f(v_i) = \{0,1,2,...,(n-1)\} \]
\[ f(u_j) = \{2n-1,2n,2n+1,...,3(n-1)\} \]
\[ f(w_j) = \{n,(n+1),...,2(n-1)\} \]

Define the induced function on edges as \( f^* : E \rightarrow N \) such that
\[ f^*(v_i u_j) = f(v_i)^2 + f(u_j)^2; \]
\[ f^*(u_j w_j) = f(u_j)^2 + f(w_j)^2; \]
\[ f^*(w_j v_{i+1}) = f(w_j)^2 + f(v_{i+1})^2 \text{ and} \]
\[ f^*(v_i v_{i+1}) = f(v_i)^2 + f(v_{i+1})^2 \]

Now,
\[ f^*(v_i u_j) = (i-1)^2 + (2(n-1)+j)^2 \]
\[ f^*(u_j w_j) = (2(n-1)+j)^2 + ((n-1)+j)^2 \]
\[ f^*(w_j v_{i+1}) = ((n-1)+j)^2 + i^2 \]
\[ f^*(v_i v_{i+1}) = (i-1)^2 + i^2 \]

To prove \( f^* \) is injective we have to prove
\[ f^*(v_i u_j) \neq f^*(v_{i+1} u_{j+1}); \]
\[ f^*(u_j w_j) \neq f^*(u_{j+1} w_{j+1}); \]
\[ f^*(w_j v_{i+1}) \neq f^*(w_{j+1} v_{i+2}) \text{ and} \]
\[ f^*(v_i v_{i+1}) \neq f^*(v_{i+1} v_{i+2}), \forall i,j \]

Now
\[ f^*(v_{i+1} u_{j+1}) = i^2 + (2n + j-1)^2 \text{ and} \]
\[ f^*(v_i u_j) = (i-1)^2 + (2(n-1)+j)^2 \]
\[ \therefore f^*(v_{i+1} u_{j+1}) \neq f^*(v_i u_j); \therefore i^2 + (2n + j-1)^2 \neq (i-1)^2 + (2(n-1)+j)^2 \]
\[ f^*(u_{j,i}w_{j,i}) = (2n + j - 1)^2 + (n + j)^2 \quad \text{and} \]
\[ f^*(u_jw_j) = (2(n - 1) + j)^2 + ((n - 1) + j)^2 \]
\[ \therefore f^*(u_{j,i}w_{j,i}) \neq f^*(u_jw_j) \quad \therefore (2n + j - 1)^2 + (n + j)^2 \neq (2(n - 1) + j)^2 + ((n - 1) + j)^2 \]
\[ f^*(w_{j,i}v_{i,j}) = (n + j)^2 + (i + 1)^2 \quad \text{and} \]
\[ f^*(w_jv_{i,j}) = ((n - 1) + j)^2 + i^2 \]
\[ \therefore f^*(w_{j,i}v_{i,j}) \neq f^*(w_jv_{i,j}) \quad \therefore (n + j)^2 + (i + 1)^2 \neq ((n - 1) + j)^2 + i^2 \]
\[ f^*(v_{i,j}v_{i,j}) = i^2 + (i + 1)^2 \quad \text{and} \]
\[ f^*(v_jv_{i,j}) = (i - 1)^2 + i^2 \]
\[ \therefore f^*(v_{i,j}v_{i,j}) \neq f^*(v_jv_{i,j}) \quad \therefore i^2 + (i + 1)^2 \neq (i - 1)^2 + i^2, \forall i, j \]

Therefore all edge labels are distinct.

Hence \( Q_n \) admits Square sum labeling.

**Theorem 2.2:**

The Quadrilateral Snake \( Q_n \) is Square difference graph for all \( n \geq 2 \).

**Proof:** Consider a path \( v_1v_2...v_n \) by joining \( v_i \) and \( v_{i+1} \) to two new vertices \( u_j \) and \( w_j, 1 \leq j \leq n-1 \)

Define a function \( f : V(Q_n) \to \{0, 1,..., 3(n-1)\} \) by

\[ f(v_i) = i - 1, 1 \leq i \leq n \]
\[ f(u_j) = 2(n - 1) + j, 1 \leq j \leq n - 1 \]
\[ f(w_j) = (n - 1) + j, 1 \leq j \leq n - 1 \]

Thus the labels of the vertices are given below:

\[ f(v_i) = \{0, 1, 2,...,(n-1)\} \]
\[ f(u_j) = \{2n-1, 2n, 2n+1,...,3(n-1)\} \]
\[ f(w_j) = \{n,(n+1),...,2(n-1)\} \]

Define the induced function on edges as \( f^* : E \to N \) such that
\[ f^*(v, u_j) = f(v_j)^2 - f(u_j)^2; \]
\[ f^*(u_j, w_j) = f(u_j)^2 - f(w_j)^2; \]
\[ f^*(w_j, v_{i+l}) = f(w_j)^2 - f(v_{i+l})^2 \]
\[ f^*(v_{i+l}, v_{i+l}) = f(v_{i+l})^2 - f(v_{i+l})^2 \]

Now,
\[ f^*(v, u_j) = (i-1)^2 - (2(n-1) + j)^2 \]
\[ f^*(u_j, w_j) = (2(n-1) + j)^2 - ((n-1) + j)^2 \]
\[ f^*(w_j, v_{i+l}) = ((n-1) + j)^2 - i^2 \]
\[ f^*(v_{i+l}, v_{i+l}) = (i-1)^2 - i^2 \]

To prove \( f^* \) is injective we have to prove

\[ f^*(v, u_j) \neq f^*(v_{i+l}, u_{i+l}); \]
\[ f^*(u_j, w_j) \neq f^*(u_{j+l}, w_{j+l}); \]
\[ f^*(w_j, v_{i+l}) \neq f^*(w_{j+l}, v_{i+l}) \text{ and} \]
\[ f^*(v_{i+l}, v_{i+l}) \neq f^*(v_{i+l}, v_{i+l}), \forall i, j \]

Now
\[ f^*(v_{i+l}, u_{i+l}) = (i^2 - 2n + j - 1)^2 \text{ and} \]
\[ f^*(v, u_j) = (i-1)^2 - (2(n-1) + j)^2 \]
\[ \therefore f^*(v_{i+l}, u_{i+l}) \neq f^*(v, u_j) \therefore i^2 - 2n + j - 1)^2 \neq (i-1)^2 - (2(n-1) + j)^2 \]
\[ f^*(u_j, w_j) = (2n + j - 1)^2 - (n + j)^2 \text{ and} \]
\[ f^*(u_{j+l}, w_{j+l}) = (2(n-1) + j)^2 + ((n-1) + j)^2 \]
\[ \therefore f^*(u_j, w_j) \neq f^*(u_{j+l}, w_{j+l}) \therefore (2n + j - 1)^2 - (n + j)^2 \neq (2(n-1) + j)^2 + ((n-1) + j)^2 \]
\[ f^*(w_j, v_{i+l}) = (n + j)^2 - (i+1)^2 \text{ and} \]
\[ f^*(w_{j+l}, v_{i+l}) = ((n-1) + j)^2 - i^2 \]
\[ \therefore f^*(w_j, v_{i+l}) \neq f^*(w_{j+l}, v_{i+l}) \therefore (n + j)^2 - (i+1)^2 \neq ((n-1) + j)^2 - i^2 \]
\[ f^*(v_{i+l}, v_{i+l}) = i^2 - (i+1)^2 \text{ and} \]
\[ f^*(v, v_{i+l}) = (i-1)^2 - i^2 \]
\[ \therefore f^*(v_{i+l}, v_{i+l}) \neq f^*(v, v_{i+l}) \therefore i^2 - (i+1)^2 \neq (i-1)^2 - i^2, \forall i, j \]

Therefore all edge labels are distinct.
Hence $Q_n$ admits Square difference labeling.

**Theorem 2.3:**

The Quadrilateral Snake $Q_n$ is Root Mean Square graph for all $n \geq 2$.

**Proof:** Consider a path $v_1v_2\ldots v_n$ by joining $v_i$ and $v_{i+1}$ to two new vertices $u_j$ and $w_j$, $1 \leq j \leq n-1$.

Define a function $f: V(Q_n) \rightarrow \{0,1,\ldots,(6n-4)\}$ by

$$
\begin{align*}
  f(v_i) &= i-1, 1 \leq i \leq n \\
  f(u_j) &= 2(n-1) + j, 1 \leq j \leq n-1 \\
  f(w_j) &= (n-1) + j, 1 \leq j \leq n-1
\end{align*}
$$

Thus the labels of the vertices are given below:

$$
\begin{align*}
  f(v_i) &= \{0,1,2,\ldots,(n-1)\} \\
  f(u_j) &= \{2n-1,2n,2n+1,\ldots,3(n-1)\} \\
  f(w_j) &= \{n,(n+1),\ldots,2(n-1)\}
\end{align*}
$$

Define the induced function on edges as $f^*: E \rightarrow N$ such that

$$
\begin{align*}
  f^*(v_i, u_j) &= \sqrt{\frac{f(v_i)^2 + f(u_j)^2}{2}} \\
  f^*(u_j, w_j) &= \sqrt{\frac{f(u_j)^2 + f(w_j)^2}{2}} \\
  f^*(w_j, v_{i+1}) &= \sqrt{\frac{f(w_j)^2 + f(v_{i+1})^2}{2}} \quad \text{and} \\
  f^*(v_i, v_{i+1}) &= \sqrt{\frac{f(v_i)^2 + f(v_{i+1})^2}{2}}
\end{align*}
$$

Now,

$$
\begin{align*}
  f^*(v_i, u_j) &= \sqrt{\frac{(i-1)^2 + (2(n-1) + j)^2}{2}} \\
  f^*(u_j, w_j) &= \sqrt{\frac{2(n-1) + j + ((n-1) + j)^2}{2}} \\
  f^*(w_j, v_{i+1}) &= \sqrt{\frac{((n-1) + j)^2 + i^2}{2}} \\
  f^*(v_i, v_{i+1}) &= \sqrt{\frac{(i-1)^2 + i^2}{2}}
\end{align*}
$$
To prove $f^*$ is injective we have to prove:

$$f^*(v_i, u_j) \neq f^*(v_i, u_{i+1})$$
$$f^*(u_j, w_j) \neq f^*(u_{j+1}, w_{j+1})$$
$$f^*(w_j, v_{i+1}) \neq f^*(w_j, v_{i+2}) \quad \text{and}$$
$$f^*(v_i, v_{i+1}) = f^*(v_i, v_{i+2}), \forall i, j$$

Now

$$f^*(v_{i+1}, u_{i+1}) = \sqrt{\frac{i^2 + (2n + j - 1)^2}{2}}$$

$$f^*(v_i, u_j) = \sqrt{\frac{(i-1)^2 + 2(n-1) + j}{2}}$$

$$\therefore f^*(v_{i+1}, u_{i+1}) \neq f^*(v_i, u_j) \because \sqrt{\frac{i^2 + (2n + j - 1)^2}{2}} \neq \sqrt{\frac{(i-1)^2 + 2(n-1) + j}{2}}$$

$$f^*(u_{j+1}, w_{j+1}) = \sqrt{\frac{2n + j - 1 + (n + j)^2}{2}}$$

$$f^*(u_j, w_j) = \sqrt{\frac{(n-1 + j)^2 + (n-1 + j)^2}{2}}$$

$$\therefore f^*(v_j, u_{i+1}) = f^*(v_j, u_{i+2}) \because \sqrt{\frac{2n + j - 1 + (n + j)^2}{2}} \neq \sqrt{\frac{(n-1 + j)^2 + (n-1 + j)^2}{2}}$$

$$f^*(w_{j+1}, v_{i+2}) = \sqrt{\frac{(n + j)^2 + (i+1)^2}{2}}$$

$$f^*(w_j, v_{i+1}) = \sqrt{\frac{(n + j)^2 + (i+1)^2}{2}}$$

$$\therefore f^*(v_{i+1}, u_{i+1}) \neq f^*(v_{i+1}, u_{i+2}) \because \sqrt{\frac{(n + j)^2 + (i+1)^2}{2}} \neq \sqrt{\frac{(n-1 + j)^2 + (i+1)^2}{2}}$$

$$f^*(v_i, v_{i+1}) = \sqrt{\frac{i^2 + (i+1)^2}{2}}$$

$$f^*(v_i, v_{i+2}) = \sqrt{\frac{(i-1)^2 + (i+1)^2}{2}}$$

$$\therefore f^*(v_{i+1}, u_{i+1}) \neq f^*(v_i, u_j) \because \sqrt{\frac{i^2 + (i+1)^2}{2}} \neq \sqrt{\frac{(i-1)^2 + (i+1)^2}{2}}, \forall i, j$$

Therefore all edge labels are distinct.
Hence $Q_n$ admits Root Mean Square labeling.

**Theorem 2.4:**

The Quadrilateral Snake $Q_n$ is Strongly Multiplicative graph for all $n \geq 2$.

**Proof:** Consider a path $v_1v_2...v_n$ by joining $v_i$ and $v_{i+1}$ to two new vertices $u_j$ and $w_j, 1 \leq j \leq n-1$

Define a function $f : V(Q_n) \rightarrow \{0,1,...,3(n-1)\}$ by

\[
\begin{align*}
    f(v_i) &= 2(n-1)+i, 1 \leq i \leq n \\
    f(u_j) &= j, 1 \leq j \leq n-1 \\
    f(w_j) &= (n-1)+j, 1 \leq j \leq n-1
\end{align*}
\]

Thus the labels of the vertices are given below:

\[
\begin{align*}
    f(v_i) &= \{2n-1,2n,2n+1,...,(3n-2)\} \\
    f(u_j) &= \{1,2,...,(n-1)\} \\
    f(w_j) &= \{n,n+1,...,2(n-1)\}
\end{align*}
\]

Define the induced function on edges as $f^* : E \rightarrow N$ such that

\[
\begin{align*}
    f^*(v_i,u_j) &= f(v_i) f(u_j) \\
    f^*(u_j,w_j) &= f(u_j) f(w_j) \\
    f^*(w_j,v_{i+1}) &= f(w_j) f(v_{i+1}) \text{ and} \\
    f^*(v_i,v_{i+1}) &= f(v_i) f(v_{i+1})
\end{align*}
\]

Now,

\[
\begin{align*}
    f^*(v_i,u_j) &= (2(n-1)+i)(j) \\
    f^*(u_j,w_j) &= j((n-1)+j) \\
    f^*(w_j,v_{i+1}) &= ((n-1)+j)(2n+i-1) \\
    f^*(v_i,v_{i+1}) &= (2(n-1)+j)(2n+i-1)
\end{align*}
\]

To prove $f^*$ is injective we have to prove
\[ f^*(v, u_j) \neq f^*(v_{i+1}, u_{j+1}); \]
\[ f^*(u, w_j) \neq f^*(u_{j+1}, w_{j+1}); \]
\[ f^*(w, v_{i+1}) \neq f^*(w_{j+1}, v_{i+2}) \quad \text{and} \]
\[ f^*(v_{i+1}, v_{i+1}) \neq f^*(v_{i+1}, v_{i+2}), \forall i, j \]

Now
\[ f^*(v_{i+1}, u_{j+1}) = (2n+i-1)(j+1) \quad \text{and} \]
\[ f^*(u, v_{j+1}) = (2n-l+i)(j) \]
\[ \therefore f^*(v_{i+1}, u_{j+1}) \neq f^*(u, v_{j+1}) \quad \because (2n+i-1)(j+1) \neq (2n-l+i)(j) \]
\[ f^*(u_{j+1}, w_{i+1}) = (j+1)(n+j) \quad \text{and} \]
\[ f^*(u_w, v_{i+1}) = j(n-l)+j \]
\[ \therefore f^*(u_{j+1}, w_{i+1}) \neq f^*(u_w, v_{i+1}) \quad \because (j+1)(n+j) \neq j(n-l)+j \]

\[ f^*(w_{j+1}, v_{i+2}) = (n+j)(2n+i) \quad \text{and} \]
\[ f^*(w, v_{i+1}) = (n-l+j)(2n+i-1) \]
\[ \therefore f^*(w_{j+1}, v_{i+2}) \neq f^*(w, v_{i+1}) \quad \because (n+j)(2n+i) \neq (n-l+j)(2n+i-1) \]
\[ f^*(v_{i+1}, v_{i+2}) = (2n+i-1)(2n+i) \quad \text{and} \]
\[ f^*(v_{i+1}, v_{i+2}) = (2n-l+i)(2n+l-1) \]
\[ f^*(v_{i+1}, v_{i+2}) \neq f^*(v_{i+1}, v_{i+2}), \forall i, j \]

Therefore all edge labels are distinct.

Hence \( Q_n \) admits Strongly Multiplicative labeling.

**Theorem 2.5:**

The Quadrilateral Snake \( Q_n \) is Cordial graph for \( n \) is odd.

**Proof:** Consider a path \( v_1 v_2 \ldots v_n \) by joining \( v_i \) and \( v_{i+1} \) to two new vertices \( u_j \) and \( w_j, 1 \leq j \leq n-1 \)

Define a function \( f : V(Q_n) \to \{0, 1\} \) by
Thus the entire (3n-2) vertices are labeled in such a way that the number of vertices labeled with ‘0’ are \( \frac{3n-1}{2} \) and the number of vertices labeled with ‘1’ are \( \frac{3n-3}{2} \).

Define the induced function on edges as \( f^*: E \to \{0,1\} \) is defined such that

\[
\begin{align*}
f^*(v_iv_{i+1}) &= |f(v_i) - f(v_{i+1})| \\
f^*(v_iu_j) &= |f(v_i) - f(u_j)| \\
f^*(u_jw_j) &= |f(u_j) - f(w_j)| \\
f^*(w_jv_{i+1}) &= |f(w_j) - f(v_{i+1})|, \forall i, j
\end{align*}
\]

Using the induced function, we see that 2(n-1) edges receive label ‘0’ and ‘1’.

Thus the entire 4(n-1) edges are labeled in such a way that the number of edges labeled ‘1’ and the number of edges labeled ‘0’ are same as 2(n-1). Thus in each cases we have \( |v_i(0) - v_j(1)| \leq 1 \) and \( |e_j(0) - e_j(1)| \leq 1 \).

Hence \( Q_n \) is Cordial.

**Theorem 2.6:**

The Quadrilateral Snake \( Q_n \) is Total Cordial graph for n is odd.

**Proof:** Consider a path \( v_iv_{i+1}...v_n \) by joining \( v_i \) and \( v_{i+1} \) to two new vertices \( u_j \) and \( w_j, 1 \leq j \leq n-1 \)

Define a function \( f: V(Q_n) \to \{0,1\} \) by
Thus the entire (3n-2) vertices are labeled in such a way that the number of vertices labeled with ‘0’ are \( \frac{3n-1}{2} \) and the number of vertices labeled with ‘1’ are \( \frac{3n-3}{2} \).

Define the induced function on edges as \( f^*: E \rightarrow \{0, 1\} \) is defined such that

\[
\begin{align*}
    f^*(v_i v_{i+1}) &= \left| f(v_i) - f(v_{i+1}) \right| \\
    f^*(v_i u_j) &= \left| f(v_i) - f(u_j) \right| \\
    f^*(u_j w_j) &= \left| f(u_j) - f(w_j) \right| \\
    f^*(w_j v_i) &= \left| f(w_j) - f(v_i) \right|, \ \forall i, j
\end{align*}
\]

Using the induced function, we see that 2(n-1) edges receive label ‘0’ and ‘1’.

Thus the entire 4(n-1) edges are labeled in such a way that the number of edges labeled ‘1’ and the number of edges labeled ‘0’ are same as 2(n-1). Thus in each cases we have \( \left| (v_i (0) + e_j (0)) - (v_i (1) + e_j (1)) \right| \leq 1 \).

Hence \( Q_n \) admits Total Cordial labeling.

**Theorem 2.7:**

The Quadrilateral Snake \( Q_n \) is Even Mean graph for all \( n \geq 2 \)

**Proof:** Consider a path \( v_1v_2...v_n \) by joining \( v_i \) and \( v_{i+1} \) to two new vertices \( u_j \) and \( w_j, 1 \leq j \leq n-1 \)

Define a function \( f: V(Q_n) \rightarrow \{2, 4, 6,..., 8(n-1)\} \) by
Thus the labels of the vertices are given below:

\[ f(v_i) = \{4, 10, 16, \ldots, (6n - 2)\} \]

\[ f(u_j) = \{2, 8, 14, \ldots, 6(n - 1) - 4\} \]

\[ f(w_k) = \{6, 12, \ldots, 6(n - 1)\} \]

Define the induced function on edges as \( f^*: E \to N \)

\[ f^*(v_i u_j) = \frac{f(v_i) + f(u_j)}{2}; \]

\[ f^*(u_j w_k) = \frac{f(u_j) + f(w_k)}{2}; \]

\[ f^*(w_k v_{i+1}) = \frac{f(w_k) + f(v_{i+1})}{2} \text{ and} \]

\[ f^*(v_i v_{i+1}) = \frac{f(v_i) + f(v_{i+1})}{2} \]

Now,

\[ f^*(v_i u_1) = 3 \]

\[ f^*(v_i u_j) = \frac{f(v_i) + f(u_{j-1}) + 12}{2}, \text{ for } 2 \leq i, j \leq n - 1 \]

\[ f^*(u_1 w_i) = 4 \]

\[ f^*(u_j w_j) = \frac{f(u_j) + f(w_{j+1}) + 12}{2}, \text{ for } 2 \leq j \leq n - 1 \]

\[ f^*(w_j v_{i+1}) = \frac{12 + f(v_i)}{2}, \text{ for } i = 1, j = 1 \]
To prove $f^*$ is injective we have to prove

$$f^*(v_i) \neq f^*(v_j),\quad f^*(u_i) \neq f^*(u_j),$$

$$f^*(w_{i,j}) \neq f^*(w_{j,i}),$$

and

$$f^*(v_{i,j}) \neq f^*(v_{j,i}), \quad \forall i, j$$

Now

$$f^*(v_{i,j}) = \frac{f(v_i) + f(u_j) + 12}{2}, \quad \text{for } 2 \leq i, j \leq n - 1$$

$$f^*(v_{i,j}) \neq f^*(v_{j,i}),$$

$$f^*(u_{i,j}) = \frac{f(u_i) + f(w_j) + 12}{2}, \quad \text{for } 2 \leq j \leq n - 1$$

$$f^*(u_{i,j}) \neq f^*(u_{j,i}),$$

$$f^*(w_{i,j}) = \frac{f(w_{i,j}) + 12 + f(v_{i,j})}{2}, \quad \text{for } 2 \leq i, j \leq n - 1$$

$$f^*(w_{i,j}) \neq f^*(w_{j,i}),$$

$$f^*(v_{i,j}) = \frac{10 + f(v_{i,j})}{2}, \quad \text{for } i = 1$$

$$f^*(v_{i,j}) \neq f^*(v_{j,i}),$$

$$f^*(v_{i,j}) = \frac{12 + f(v_{i,j}) + f(v_{j,i})}{2}, \quad \text{for } 2 \leq i \leq n - 1$$

$$f^*(v_{i,j}) \neq f^*(v_{j,i}), \quad \forall i, j$$

Therefore all edge labels are distinct.
Hence $Q_n$ admits Even Mean labeling.

**Theorem 2.8:**

The Quadrilateral Snake $Q_n$ is Odd Mean graph for all $n \geq 2$

**Proof:** Consider a path $v_1v_2...v_n$ by joining $v_i$ and $v_{i+1}$ to two new vertices $u_j$ and $w_j, 1 \leq j \leq n-1$

Define a function $f : V(Q_n) \rightarrow \{1, 3, 5, ..., (8n-9)\}$ by

- $f(v_i) = 3$, for $i = 1$
- $f(v_i) = f(v_{i+1}) + 6$, for $2 \leq i \leq n$
- $f(u_j) = 1$, for $j = 1$
- $f(u_j) = f(u_{j+1}) + 6$, for $2 \leq j \leq n-1$
- $f(w_j) = 5$, for $j = 1$
- $f(w_j) = f(w_{j+1}) + 6$, for $2 \leq k \leq n-1$

Thus the labels of the vertices are given below:

- $f(v_i) = \{3, 9, 15, ..., (6n-3)\}$
- $f(u_j) = \{1, 7, 11, ..., 6(n-1)-5\}$
- $f(w_j) = \{5, 11, ..., 6(n-1)-1\}$

Define the induced function on edges as $f^* : E \rightarrow N$

- $f^*(v_iu_j) = \frac{f(v_i) + f(u_j)}{2}$
- $f^*(u_jw_j) = \frac{f(u_j) + f(w_j)}{2}$
- $f^*(w_jv_{i+1}) = \frac{f(w_j) + f(v_{i+1})}{2}$ \text{ and}
- $f^*(v_{i+1}v_i) = \frac{f(v_{i+1}) + f(v_i)}{2}$

Now,
To prove $f^*$ is injective we have to prove

$$f^*(v_i u_j) \neq f^*(v_i u_{j+1});$$
$$f^*(u_i w_j) \neq f^*(u_i w_{j+1});$$
$$f^*(w_j v_{i+1}) \neq f^*(w_j v_{i+2})$$ and

$$f^*(v_i v_{i+1}) \neq f^*(v_i v_{i+2}), \forall i, j$$

Now

$$f^*(v_i u_{i+1}) = \frac{f(v_i) + f(u_i) + 12}{2}, \text{ for } 2 \leq i, j \leq n - 1$$
$$f^*(v_i u_{i+1}) \neq f^*(v_i u_j)$$
$$f^*(u_i w_{i+1}) = \frac{f(u_i) + f(w_i) + 12}{2}, \text{ for } 2 \leq j \leq n - 1$$
$$f^*(u_i w_{i+1}) \neq f^*(u_i w_j)$$

$$f^*(w_i v_{i+1}) = \frac{11 + f(v_i)}{2}, \text{ for } i, j = 1$$
$$f^*(w_i v_{i+1}) \neq f^*(w_i v_{i+2})$$

$$f^*(w_i v_{i+1}) = \frac{f(w_i) + f(v_{i+1})}{2}, \text{ for } 2 \leq i, j \leq n - 1$$
$$f^*(w_i v_{i+1}) \neq f^*(w_i v_{i+2})$$

$$f^*(w_i v_{i+1}) = \frac{f(w_i) + f(v_{i+1})}{2}, \text{ for } 2 \leq i, j \leq n - 1$$
$$f^*(w_i v_{i+1}) \neq f^*(w_i v_{i+2})$$
Therefore all edge labels are distinct.

Hence $Q_n$ admits Odd Mean labeling.

3. Conclusion

In this paper we have examined the existence of square sum, square difference, Root Mean square, Strongly Multiplicative, Odd mean, Even mean, Cordial and Total Cordial labeling for Quadrilateral snake graph $Q_n$ for all $n \geq 2$. Further investigation can be done to obtain the above labeling for some class of graphs.

References


