THE EXISTENCE AND
UNIQUENESS OF THE SOLUTION
FOR THE STOCHASTIC INTEGRO
DIFFERENTIAL EQUATIONS WITH
INFINITE DELAY

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Abstract
We consider the following stochastic integro-differential equation with infinite delay of the form

\[ dx(t) = \left[ f(t, x_t) + \int_0^t g(t, s, x_s)ds \right] dt + \sigma(t, x_t)dB(t), 0 \leq t \leq T \]

with initial condition \( x_0 = \xi = \{ \xi(\theta) : -\infty < \theta \leq 0 \} \) is \( f_0 \) measurable, where \( x_t = \{ x(t + \theta) : -\infty < \theta \leq 0 \} \) can be regarded as a \( BC((-\infty, 0]; \mathbb{R}^d) \) value stochastic process, \( f, \sigma \) and \( g \) be a Borel measurable. We prove the existence and uniqueness of the above equation by using Picard’s successive approximation. We also study the error between approximate solution and accurate solution.

Key Words: Fixed point theorem, Integro-differential equation, Picard’s successive approximation.
1 Introduction

System in many branches of science and industry are often perturbed by various types of environment noise. For consider the following simple population growth model.

\[
\frac{dN(t)}{dt} = a(t)N(t)
\]  

(1.1)

with the initial value \(N(0) = N_0\), where \(N(t)\) is the size of population at time \(t\) and \(a(t)\) is the relative rate of growth. It might happen that \(a(t)\) is not completely known, but subject to some random environmental effects.

In other words, each size of population is moving in deterministic path, the motion of a collection of them is computationally and practically unpredictable. A large enough set of size of population will exhibit stochastic characteristics such as death due to natural and artificial calamities. These are emergent properties of the system. Now, in general

\[
a(t) = r(t) + \sigma(t)'noise'
\]

so equation (1.1) becomes

\[
\frac{dN(t)}{dt} = r(t)N(t) + \sigma(t)N(t)'noise'
\]

or equivalently,

\[
N(t) = N_0 + \int_0^t r(s)N(s)ds + \int_0^t \sigma(s)N(s)'noise'ds.
\]  

(1.2)

It turn out that a reasonable mathematical interpretation for the 'noise' term is the so called white noise \(B(t)\), which is formally regarded as the derivative of a Brownian motion \(B(t)\), i.e., \(B(t) = frac{dB(t)}{dt}\). So, the term 'noise' \(dt\) can be expressed as \(B(t)dt = dB(t)\), and

\[
\int_0^t \sigma(s)N(s)'noise'ds = \int_0^t \sigma(s)N(s)dB(s).
\]  

(1.3)
The simple stochastic population growth model

\[ N(t) = N_0 + \int_0^t r(s)N(s)ds + \int_0^t \sigma(s)N(s)dB(s) \]

or in the differential form,

\[ dN(t) = r(t)N(t)dt + \sigma(t)N(t)dB(t), \quad t \geq 0. \quad (1.4) \]

Stochastic differential equation are used in a wide range of application in environmental modeling, engineering and biological modeling. The typically describe the time dynamics of the evolution of a state vector, based on the (approximate) physics of the real system, together with a during noise process. The noise process can be thought of in present several ways. It often represent process not include in the model, but present in the real system. So, it is important to study the stochastic differential equation.

Mao Xuerong [8], has studied the stochastic differential equation of the form

\[ dX(t) = f(X(t), t)dt + g(X(t), t)dB(t), \quad t_0 \leq t \leq T \quad (1.5) \]

with initial value \( X(t_0) = X_0 \). This is equivalent to the following stochastic integral equation of the form

\[ X(t) = X_0 + \int_{t_0}^t f(X(s), s)ds + \int_{t_0}^t g(X(s), s)dB(s), \quad t_0 \leq t \leq T. \quad (1.6) \]

He used Lipschitz condition (1.7) Linear growth condition (1.8) as follows

\[ |f(X, t) - f(X, t)|^2 \vee |g(X, t) - g(X, t)|^2 \leq K|X - X|^2, \quad K > 0 \quad (1.7) \]

for any \((X, t) \in \mathcal{D} \times [t_0, T] \) follows that

\[ |f(X, t)|^2 \vee |g(X, t)|^2 \leq K(1 + |X|^2), \quad K > 0 \quad (1.8) \]

then (1.5) had a unique solution \( X(t) \), moreover \( X(t) \in M^2([t_0, T]; \mathcal{D}). \)
T. Taniguchi [11], studied existence and uniqueness of the equation (1.5) by reducing Lipschitz and growth condition by a generalize Holder type function on $f$ and $g$. Furthermore, mao [8] also discussed the stochastic functional differential equation with finite delay.

$$dX(t) = f(X_t, t)dt + g(X_t, t)dB(t), t_o \leq t \leq T$$ (1.9)

where $X_t = \{X(t + \theta) : -\tau \leq \theta \leq 0\}$. Could be considered as a $C([-\tau, 0]; \mathcal{D})$-value stochastic process, the initial value of (1.7) was proposed as follows $X_{t_o} = \xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$ is an $f_{t_o}$-measurable, $C([-\tau, 0]; \mathcal{D})$-value random such that

$$E \|\xi\|^2 < \infty.$$ (1.10)

From system (1.9), if uniform Lipscitz condition (1.11) and linear growth condition (1.12) are satisfied, that is, for any $\varphi, \psi \in C([-\tau, 0]; \mathcal{D})$ and $t \in [t_0, T]$, it follows that

$$|f(\varphi, t) - f(\psi, t)|^2 \vee |g(\varphi, t) - g(\psi, t)|^2 \leq K|\varphi - \psi|^2, K > 0$$ (1.11)

for any $(\varphi, t) \in C([-\tau, 0]; \mathcal{D}) \times [t_0, T]$ it follows that

$$|f(\varphi, t)|^2 \vee |g(\varphi, t)|^2 \leq K(1 + \|\varphi\|), K > 0$$ (1.12)

then (1.9) had a unique solution $X(t)$, moreover, $X(t) \in M^2([t_0 - \tau, T]; \mathcal{D})$. Many authors [5,6,7,12,13] has been studied the existence and uniqueness results for stochastic differential equation and stochastic functional differential equation. The related results of existence, uniqueness and stability of solution could also be found in the literature [1,2,3,4,9,10,13].

Weifengying and Wang Ke [12] extended [8] and proved the existence and uniqueness of solution for stochastic functional differential equation with infinite delay of the form

$$dX(t) = f(X_t, t)dt + g(X_t, t)dB(t), t \in [t_0, \infty).$$

Moreover he studied the estimate for the error between solution and accurate solution.
In this paper, we study the existence and uniqueness theorem of the solution for stochastic integro-differential equation of the form

$$dx(t) = [f(t, x_t) + \int_0^t g(t, s, x_s)ds]dt + \sigma(t, x_t)dB(t), 0 \leq t \leq T$$

with initial condition $x_0 = \xi = \{\xi(\theta) : -\infty < \theta \leq 0\}$ is $f_0$ measurable, where $x_t = \{x(t + \theta) : -\infty < \theta \leq 0\}$ can be regarded as a $BC((-\infty, 0]; \mathcal{B}^d)$ value stochastic process, $f, \sigma$ and $g$ be a Borel measurable.

2 Preliminaries

Probability theory deal with mathematical model of trials whose outcomes depend on chance. All the possible outcomes the elementary events are grouped together to form a set $\Omega$ with typical element $\omega \in \Omega$. Not every subset of $\Omega$ is a general observable or interesting events together as a family $f$ of subset of $\Omega$.

For the purpose of probability theory, such a family $f$ should have the following properties:

(i). $\varnothing \in f$, where $\varnothing$ denotes the empty set;

(ii). $A \in f \Rightarrow A^C \in f$, where $A^C = \Omega - A$ is the complement of $A$ in $\Omega$;

(iii). $\{A_i\}_{i \geq 1} \subset f \Rightarrow \bigcup_{i \geq 1} A_i \in f$.

A family $f$ with these three properties is called a $\sigma$-algebra. The pair $(\Omega, f)$ is called a measurable space and the elements of $f$ is hence worth called $f$-measurable sets instead of events. If $c$ is a family subsets of $\Omega$, then there exists a smallest $\sigma$-algebra $\sigma(c)$ on $\Omega$ which contains $c$. This $\sigma(c)$ is called the $\sigma$-algebra generated by $c$. If $\Omega = \mathcal{B}^d$ and $c$ is the family of the open sets in $\mathcal{B}^d$, then $\mathcal{B}^d = \sigma(c)$ is called Borel $\sigma$- algebra and the elements of $\mathcal{B}^d$ are called the Borel sets.

**Stochastic Process:**

Let $(\Omega, f, P)$ be the probability space. A filtration is a family $\{f_t\}_{t \geq 0}$ of increasing sub $\sigma$-algebra of $f$ i.e., $f_t \subset f_s \subset f$ for all $0 \leq t < s < \infty$.

The filtration is said to be right continuous if $f_t = \bigcap_{s>t} f_s$ for all $t \geq 0$. When the probability space is complete, the filtration
is said to satisfy the usual conditions if it is right continuous and $f_0$ contains all $P$-null sets. A family $\{X_t\}_{t \in I}$ of $\mathcal{F}^d$-valued random variables is called a stochastic process with parameter set (or index set) $I$ and state space $\mathcal{F}^d$. A stochastic process, or some times random process, is the counter part of a deterministic process (or deterministic system) in probability theory.

**Brownian Motion:**
Brownian motion is the name given to the irregular movement of pollen grains, suspended in water, observed by the Scottish botanist Robert Brown in 1828. The motion was later explained by the random collisions with the molecules of water. To describe the motion mathematically it is natural to use the concept of a stochastic process $B_t(\omega)$, interpreted as the pollen grain $\omega$ at time $t$. Mathematical definition of Brownian motion in (2.4)

If $\{B_t\}_{t \geq 0}$ is a Brownian motion and $0 \leq t_0 < t_1 < \ldots < t_k < \infty$, then the increments $B_{t_i} - B_{t_{i-1}}, 1 \leq i \leq k$ are independent and we say that the Brownian motion has independent increments. Moreover, the distribution of $B_{t_i} - B_{t_{i-1}}$ depends only on the difference $t_i - t_{i-1}$ and we say that the Brownian motion has stationary increments.

The Brownian motion has many impotent properties and some of them are summarized below.
(a). $\{-B_t\}$ is a Brownian motion with respect to the same filtration $\{f_t\}$.
(b). Let $C > 0$. Define $\frac{B_{tC}}{\sqrt{C}}$ for $t \geq 0$.
Then $\{X_t\}$ is a Brownian motion with respect to the filtration $\{fCt\}$.
(c). $\{B_t\}$ is continuous square-integrable martingale and its quadratic variation $\langle B, B_t \rangle$ or all $t \geq 0$.
(d). The strong law of large numbers state that $\lim_{t \to \infty} \frac{B_t}{t} = 0$ a.s.
(e). For almost every $\omega \in \Omega$, the Brownian motion sample $B(\omega)$ is nowhere differentiable.

**Definition 1.** Let $\Omega$ be a non empty set. A $\sigma$-field $f$ on $\Omega$ is a family subsets of $\Omega$ such that
(1). The empty set $\emptyset$ belongs to $f$.
(2). If $A$ belongs to $f$, then so does complement $\Omega / A$.
(3). If $A_1, A_2, \ldots$ is a sequence of sets in $f$, then their union $A_1 \cup A_2 \cup \ldots$ also belongs to $f$. 

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Example 2. \( \mathbb{R} \) will denote the set of real numbers. Thus family of Borel sets \( f = B(\mathbb{R}) \) is a \( \sigma \)-field on \( \mathbb{R} \). We recall that \( B(\mathbb{R}) \) is the smallest \( \sigma \)-field containing all interval in \( \mathbb{R} \).

Definition 3. Let \( f \) be a \( \sigma \)-field on \( \Omega \). A probability measure \( P \) is a function \( P: f \to [0, 1] \) such that

1. \( P(\Omega) = 1 \).
2. If \( A_1, A_2, \ldots \) are pairwise disjoint sets (i.e., \( A_i \cap A_j = \emptyset \) for \( i \neq j \)) belongs to \( f \), then also belongs to \( f \), then \( P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots \).

The triple \( (\Omega, f, P) \) is called a probability space. The set belonging to \( f \) are called events. An event \( A \) is said to be occur almost surely (a.s) whenever \( P(A) = 1 \).

Definition 4. A sequence of \( \sigma \)-field \( f_1, f_2, \ldots \) on \( \Omega \) such that \( f_1 \subset f_2 \subset \ldots \subset f \) is called filtration.

Definition 5. Let \( (\Omega, f, P) \) be a probability space with filtration \( \{f_t\}_{t \geq 0} \). A one dimensional Brownian motion is a real-valued continuous \( f_t \)-adapted \( \{B_t\}_{t \geq 0} \) with the following properties

(i). \( B_0 = 0 \) a.s
(ii). For \( 0 \leq s < t < \infty \), the increment \( B_t - B_s \) is normally distributed with mean zero and variance \( t - s \).
(iii). For \( 0 \leq s < t < \infty \), the increment \( B_t - B_s \) is independent of \( f_s \).

Definition 6. A random variable \( \tau \) with values in the set \( \{1, 2, \ldots \} \cup \{\infty\} \) is called stopping time (with respect to filtration \( f_n \)) if for each \( n = 1, 2, \ldots \) \( \{\tau = n \in f_n\} \).

As the time \( n \) increases, so does our knowledge about what has happened in the past. This can be modeled by a filtration as defined above.

Definition 7. Gronwall Inequality:
Let \( T > 0 \) and \( C \geq 0 \). Let \( u(.) \) be a Borel measurable bounded non-negative function on \([0, T] \), and let \( V(.) \) be a non-negative integrable function on \([0, T] \). If \( u(t) \leq C + \int_0^t V(s)u(s)ds \) for all \( 0 \leq t \leq T \),
then \( u(t) \leq C \exp \left( \int_0^t V(s)ds \right) \) for all \( 0 \leq t \leq T \).
Definition 8. Holder’s Inequality:

\[ |E(X^T Y)| \leq (E|X|^p)^{\frac{1}{p}} (E|Y|^q)^{\frac{1}{q}} \]

if \( p > 1, \frac{1}{p} + \frac{1}{q} = 1, X \in L^p, Y \in L^q \).

Definition 9. Chebyshev’s inequality:

\[ P\{ \omega : |X(\omega)| \geq C \} \leq C^{-p} E|X|^p \]

A sample application of Holder’s inequality implies \((E|X|^r)^{\frac{1}{r}} \leq (E|X|^q)^{\frac{1}{q}}\) if \( 0 < r < p < \infty, X \in L^p \).

Lemma 10. if \( \sum_{n=1}^{\infty} P(A_n) \), then \( \sum_{n=1}^{\infty} P(A_n, i.o) = 0 \). [where i.o= infinitely often]

Lemma 11. If \( P \geq 2, g \in (M^2[0,T]; \mathbb{R}^{d\times m}) \) such that

\[ E\int_0^T |g(s)|^p ds < \infty \]

\[ E\int_0^T |g(s)dB(s)|^p \leq \left[ \frac{p(p-1)}{2} \right]^{\frac{1}{2}} T^{\frac{p-2}{2}} E\int_0^T |g(s)|^p ds, \]

Mao had shown Lemma 11 in [8,page:39]

3 Existence Results

In this paper, we prove the existence and uniqueness of the stochastic integro-differential equation of the form

\[ dx(t) = [f(t, x_t) + \int_0^t |g(t, s, x_s)ds]dt \]

\[ \sigma(t, x_t)dB(t), 0 \leq t \leq T \]  \hspace{1cm} (3.1)

with initial condition

\[ x_0 = \xi = \{ \xi(\theta) : -\infty < \theta \leq 0 is \text{measurable} \} \]  \hspace{1cm} (3.2)
where \( x_t = \{x(t + \theta) : -\infty < \theta \leq 0\} \) can be regarded as a \( BC((-\infty, 0]; \mathbb{R}^d) \) value stochastic process, and

\[
\begin{align*}
    f : & BC((-\infty, 0]; \mathbb{R}^d) \times [0, T] \to \mathbb{R}^d \\
    \sigma : & BC((-\infty, 0]; \mathbb{R}^d) \times [0, T] \to \mathbb{R}^{d \times m} \\
    g : & BC((-\infty, 0]; \mathbb{R}^d) \times [0, T] \to \mathbb{R}^d
\end{align*}
\]

be a borel measurable.

**Definition 12.** \( \mathbb{R}^d \)-valued stochastic process \( x(t) \) defined on \( -\infty < t \leq T \) is called the solution of (3.1) with initial (3.2), if \( x(t) \) has the following properties:

(i). \( x(t) \) is continuous and \( \{x(t)\}_{0 \leq t \leq T} \) is \( f_0 \)-adapted.

(ii). \( \{f(t, x(t))\} \in L^2([0, T] : \mathbb{R}^d) \), \( \{\sigma(t, x(t))\} \in L^1([0, T] : \mathbb{R}^{d \times m}) \) and \( \{g(t, s, x(s))\} \in L^1([0, T] : \mathbb{R}^d) \)

(iii). \( x_0 = \xi \) for \( 0 \leq t \leq T \).

\[
\begin{align*}
    x(t) = & \xi(0) + \int_0^t [f(s, x(s)) + \int_0^s g(s, u, x(u))du]ds + \int_0^t \sigma(s, x(s))dB(s), \\
    x(t) = & \xi(0) + \int_0^t f(s, x(s))ds + \int_0^t \int_0^s g(s, u, x(u))du]ds + \int_0^t \sigma(s, x(s))dB(s)
\end{align*}
\]

\( x(t) \) is called as a unique solution, if any other solution \( \bar{x}(t) \) is distinguishable with \( x(t) \), that is

\[
P(x(t) = \bar{x}(t), forany - \infty < t \leq T) = 1.
\]

Let \((\Omega, f, P)\) be a complete probability space with a filtration \( \{f_t\}_{t \geq 0} \) satisfying the usual conditions i.e., it is a right continuous and \( f_0 \) contains all \( P \)-null sets. Assume that \( B(t) \) is an \( m \)-dimensional Brownian motion defined on complete probability space, that is

\[
    B(t) = (B_1(t), B_2(t), \ldots, B_m(t))^T.
\]

Let \( BC((-\infty, 0]; \mathbb{R}^d) \) denote the family of bounded continuous \( \mathbb{R}^d \)-value function \( \varphi \) defined on \((-\infty, 0]\) with norm

\[
    \|\varphi\| = \sup_{-\infty < \theta \leq 0} |\varphi(\theta)|
\]
At first glance, we derive that initial data of stochastic process must define on \((-\infty,0]\). So, the initial value is followed
\[
x_0 = \xi = \{\xi(\theta) : -\infty < \theta \leq 0 \text{ is } f_0 - \text{measurable}\}
\]

Now we assume the following hypothesis:

\(H_1\): The function \(f, g\) and \(\sigma\) satisfies the Lipschitz condition and existing constant \(L_1 > 0\) for \(x_1, x_2 \in \mathbb{R}\) and \(0 \leq s < t \leq T\) such that
\[
\|f(t, x_1) - f(t, x_2)\|^2 + \|\sigma(t, x_1) - \sigma(t, x_2)\|^2 \leq L_1 \|x_1 - x_2\|^2
\]

\(H_2\): The function \(f, g\) and \(\sigma\) are continuous and satisfies the usual linear growth condition, that is, there exists a constant \(L_2 > 0\) such that
\[
\|f(t, x)\|^2 + \|g(t, s, x(s))\|^2 + \|\sigma(t, x)\|^2 \leq L_2(\|x\|^2 + 1)
\]
for all \(t \in [0, T]\) and all \(x \in \mathbb{R}^d\).

**Lemma 13.** Let \((H_3)\) holds, if \(x(t)\) is the solution of \((3.1)\) with initial data \((3.2)\) then
\[
\mathbb{E}(\sup_{-\infty < t \leq T} |x(t)|^2) \leq \mathbb{E}\|\xi\|^2 + Ce^{4L_2T(1+2T)}
\]
where \(C = 4\mathbb{E}\|\xi\|^2 + 4L_2T(1+2T)\). In addition, \(x(t) \in \mathcal{M}^2((-\infty, T]; \mathbb{R}^d)\).

**Proof.** Let for each number \(n \geq 1\), defined the stopping time
\[
\tau_n = T \land \inf\{t \in [0, T) : \|x(t)\| \geq n\}
\]

obviously, as \(n \to \infty, \tau_n \uparrow T\) a.s.

Let \(x^n(t) = x(t \land \tau_n), t \in [0, T]\), then \(x^n(t)\) satisfy the following equation
\[
x^n(t) = \xi(0) + \int_0^t f(s, x^n_u)ds + \int_0^t \int_0^s g(s, u, x^n_u)du\,ds + \int_0^t \sigma(s, x^n_u)dB(s),
\]
\[
x^n(t) = \xi(0) + \int_0^t f(s, x^n_{\tau_n})I_{[0, \tau_n]}(s)ds + \int_0^t \int_0^s g(s, u, x^n_u)I_{[0, \tau_n]}(s)du\,ds
\]
\[
+ \int_0^t \sigma(s, x^n_{\tau_n})I_{[0, \tau_n]}(s)dB(s)
\]

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By the elementary inequality \((a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2)\), one gets that

\[
|\dot{x}(t)|^2 = |\xi(0) + \int_0^t f(s, \dot{x}_s) I_{[0,\tau_n]}(s) ds + \int_0^t \int_0^s g(s, u, \dot{x}_u) du I_{[0,\tau_n]}(s) ds \\
+ \int_0^t \sigma(s, \dot{x}_s) I_{[0,\tau_n]}(s) dB(s)|^2 \\
\leq 4|\xi(0)|^2 + 4\int_0^t f(s, \dot{x}_s) I_{[0,\tau_n]}(s) ds|^2 + 4 \int_0^t \int_0^s g(s, u, \dot{x}_u) du I_{[0,\tau_n]}(s) ds|^2 \\
+ 4\int_0^t \sigma(s, \dot{x}_s) I_{[0,\tau_n]}(s) dB(s)|^2
\]

taking expectation on both sides, we get

\[
E|\dot{x}(t)|^2 \leq 4E|\xi|^2 + 4TL_2 \int_0^t E\|\dot{x}_s\|^2 ds + 4T^2 L_2
\]

\[
+ 4TL_2 \int_0^t E\|\dot{x}_s\|^2 ds + 4T^2 L_2 + 4L_2 \int_0^t E\|\dot{x}_s\|^2 ds + 4T^2 L_2
\]

\[
E|\dot{x}(t)|^2 \leq 4E|\xi|^2 + 4L_2 T(1 + 2T) + 4L_2 (1 + 2T) \int_0^t E\|\dot{x}_s\|^2 ds
\]

\[
E(\sup_{0 \leq s \leq t} E|\dot{x}(s)|^2) \leq 4E|\xi|^2 + 4TL_2 \int_0^t E\|\dot{x}_s\|^2 ds + 4T^2 L_2
\]

\[
+ 4TL_2 \int_0^t \sup_{0 \leq s \leq t} E|\dot{x}(r)|^2 ds
\]

\[
\leq C + 4TL_2 \int_0^t \sup_{0 \leq s \leq t} E|\dot{x}(r)|^2 ds
\]

where \(C = 4E|\xi|^2 + 4L_2 T(1 + 2T)\). By Gronwall inequality, it follows that \(E(\sup_{0 \leq s \leq t} E|\dot{x}(s)|^2) \leq Ce^{4L_2 T(1 + 2T)}, 0 \leq t \leq T\) noting to
fact that
\[
\sup_{-\infty < s \leq t} |x^n(s)|^2 \leq \|\xi\|^2 + \sup_{0 \leq s \leq t} |x^n(s)|^2
\]
\[
E(\sup_{-\infty < s \leq t} |x^n(s)|^2) \leq E\|\xi\|^2 + Ce^{4L_2T(1+2T)}
\]
consequently,
\[
E(\sup_{-\infty < s \leq t} |x(s)|^2) \leq E\|\xi\|^2 + Ce^{4L_2T(1+2T)}
\]
leaving \(n \to \infty\), it then implies the following inequality,
\[
E(\sup_{-\infty < s \leq T} |x(s)|^2) \leq E\|\xi\|^2 + Ce^{4L_2T(1+2T)}
\]
This shows that the solution \(x(t)\) is uniformly bounded in \(M((-\infty, T]; \mathbb{R}^d)\).

**Theorem 14.** Assume that the condition (H1), (H2) and (H3) hold. The initial value problem (3.1) – (3.2) has a unique solution \(x(t)\). Moreover, \(x(t) \in \mathcal{M}((-\infty, T]; \mathbb{R}^d)\).

**Proof.** Define \(x^0 = \xi\) and \(x^0(t) = \xi(0)\), for \(0 \leq t \leq T\). Let \(x^n(t) = \xi, n = 1, 2, \ldots\) and define Picard sequence
\[
x^n(t) = \xi(0) + \int_0^t f(s, x^{n-1}_s)ds + \int_0^t \int_0^s g(s, u, x^{n-1}_u)du ds
\]
\[+ \int_0^t \sigma(s, x^{n-1}_s)dB(s), 0 \leq t \leq T \]
(3.4)

Obviously, \(x^0(t) \in \mathcal{M}((-\infty, T]; \mathbb{R}^d)\) by induction \(x^n(t) \in \mathcal{M}((-\infty, T]; \mathbb{R}^d)\), in fact,
\[
|x^n(t)|^2 = |\xi(0) + \int_0^t f(s, x^{n-1}_s)ds + \int_0^t \int_0^s g(s, u, x^{n-1}_u)du ds
\]
\[+ \int_0^t \sigma(s, x^{n-1}_s)dB(s)|^2
\]

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\[ \leq 4|\xi(0)|^2 + 4 \int_0^t |f(s, x_{n-1}^s)|^2 ds + 4 \int_0^t \left[ \int_0^s |g(s, u, x_{n-1}^u)| du \right] ds \]

\[ + 4 \int_0^t \sigma(s, x_{n-1}^s) dB(s)^2 \]

taking expectation on both sides, we get

\[ E|x^n(t)|^2 \leq 4E \|\xi\|^2 + 4L_2T(1 + 2T) + 4L_2(1 + 2T) \int_0^t E \|x_{n-1}^s\|^2 ds \]

\[ \leq C + 4L_2(1 + 2T) \int_0^t |x_{n-1}^s|^2 ds \]

where \( C = 4E \|\xi\|^2 + 4L_2T(1 + 2T) \). Hence for any \( k \geq 1 \), one can derive that

\[ \max_{1 \leq n \leq k} E|x^n(t)|^2 \leq C + 4L_2(1 + 2T) \int_0^t \max_{1 \leq n \leq k} \|x_{n-1}^s\|^2 ds \]

\[ \max_{1 \leq n \leq k} E|x^n_s|^2 = \max \left\{ E|\xi(0)|^2, E \|x_1^1\|^2, \ldots, E \|x_{k-1}^k\|^2 \right\} \]

\[ \leq \max \left\{ E|\xi(0)|^2, E \|x_1^1\|^2, \ldots, E \|x_{k-1}^k\|^2, E \|x_k^k\|^2 \right\} \]

\[ = \max \left\{ E|\xi(0)|^2, \max_{1 \leq n \leq k} E|x^n(s)|^2 \right\} \]

\[ \leq 4E|\xi|^2 + \max_{1 \leq n \leq k} E|x^n_s|^2 \]

\[ \max_{1 \leq n \leq k} E|x_{n-1}^s|^2 \leq E|\xi|^2 + \max_{1 \leq n \leq k} E|x^n(s)|^2 \]

\[ \max_{1 \leq n \leq k} E|x^n(t)|^2 \leq C + 4L_2(1 + 2T) \int_0^t [E \|\xi\|^2 + \max_{1 \leq n \leq k} E|x^n_s|^2] ds \]

\[ \leq C_1 + 4L_2(1 + 2T) \int_0^t \max_{1 \leq n \leq k} E|x^n_s|^2 ds \]
Where \( C_1 = C + 4L_2(1 + 2T) \int_0^t E \| \xi \|^2 \, ds \).

From Gronwall’s inequality one gets that
\[
\max_{1 \leq n \leq k} E|\text{x}^n(t)|^2 \leq C_1 e^{4L_2(1+2T)t}
\]
since \( k \) is arbitrary,
\[
E|\text{x}^n(t)|^2 \leq C_1 e^{4L_2(1+2T)t}, \quad 0 \leq t \leq T
\]

Now \( x^0(t) = \xi \)
\[
x^n(t) = \xi(0) + \int_0^t f(s, x_{n-1}^s)ds + \int_0^t \int_0^s g(s, u, x_{n-1}^{u-})du\,ds + \int_0^t \sigma(s, x_{n-1}^s)dB(s)
\]

\[
|x^1(t) - x^0(t)|^2 = \left| \int_0^t f(s, x_0^s)ds + \int_0^t \int_0^s g(s, u, x_0^{u-})du\,ds + \int_0^t \sigma(s, x_0^s)dB(s) \right|^2
\]

\[
\leq 3\|f(0, x_0^0)\|^2 + 3\|g(0, x_0^0, 0)\|^2 + 3\|\sigma(0, x_0^0)\|^2
\]

taking expectation on both sides,
\[
E|x^1(t) - x^0(t)|^2 \leq 3L_2T(1 + 2T) + 3L_2(1+2T) \int_0^t \|x_0^0\|^2 \, ds
\]
\[
E|x^1(t) - x^0(t)|^2 \leq 3L_2T(1 + 2T) + 3L_2(1+2T)E \|\xi\|^2
\]

By the same way, we compute
\[
|x^2(t) - x^1(t)|^2 = \left| \int_0^t \{f(s, x_1^s) - f(s, x_0^s)\}ds + \int_0^t \int_0^s \{g(s, u, x_1^{u-})du - g(s, u, x_0^{u-})du\}\,ds + \int_0^t \{\sigma(s, x_1^s) - \sigma(s, x_0^s)\}dB(s) \right|^2
\]

\[
\leq 3L_1(1 + 2T) \int_0^t E \|x_1^s - x_0^s\|^2 \, ds \leq ME \int_0^t E \|x_1^s - x_0^s\|^2 \, ds
\]
where $M = 3L_1(1 + 2T)$. Thus, we derive that

$$E(\sup_{0 \leq s \leq t} |x^2(s) - x^1(s)|^2) \leq ME \int_0^t \|x^1_s - x^0_s\|^2 ds \leq MTC$$

where $M = 3L_1(1 + 2T)$.

$$E(\sup_{0 \leq s \leq t} |x^3(s) - x^2(s)|^2) \leq M \int_0^t E(\sup_{0 \leq r \leq s} |x^2(r) - x^1(r)|^2) ds$$

$$\leq M \int_0^t MsCds \leq \frac{C(MT)^2}{2}$$

Now claim that for all $n \geq 0$

$$E(\sup_{0 \leq s \leq t} |x^{n+1}(s) - x^n(s)|^2) \leq \frac{C(MT)^n}{n!}, 0 \leq t \leq T. \quad (3.5)$$

When $n = 0, 1, \ldots$ inequality (3.5) holds for some $n$, now to check (3.5) for $n + 1$

$$E(\sup_{0 \leq s \leq t} |x^{n+2}(s) - x^{n+1}(s)|^2) \leq M \int_0^t E(\sup_{0 \leq r \leq s} |x^{n+1}(r) - x^n(r)|^2) ds$$

$$\leq M \int_0^t \frac{C(MT)^n}{n!} ds \leq \frac{C(MT)^{n+1}}{n + 1!}.$$ 

it is easy to see that (3.5) holds for $n + 1$. By induction (3.5) holds for $n \geq 0$. Next to verify that \{x^n(t)\} converge to $x(t)$ at the same of $L^2$ and probability 1 on $\mathcal{M}^2((-\infty, T]; \mathbb{R}^d)$. $x(t)$ is the solution of (3.1) with initial data (3.2) for (3.5) taking $t = T$.

$$E(\sup_{0 \leq t \leq T} |x^{n+1}(t) - x^n(t)|^2) \leq \frac{C(MT)^n}{n!}.$$ 

By the Chebyshev inequality,

$$P(\sup_{0 \leq t \leq T} |x^{n+1}(t) - x^n(t)|^2 > \frac{1}{2^n}) \leq \frac{C(4MT)^n}{n!}.$$ 

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from the fact $\sum_{n=0}^{\infty} \frac{C(4MT)^n}{n!} < \infty$

$$|x^{n+1}(t) - x^n(t)|^2 \leq \frac{1}{2^n}a_n \geq \eta_0$$

$$x^0(t) + \sum_{n=1}^{\infty} [x^n(t) - x^{n-1}(t)] = x^n(t)$$

is the partial sum of function series

$$x^0(t) + [x^1(t) - x^0(t)] + \ldots + [x^n(t) - x^{n-1}(t)] + \ldots$$ -----(3.6)

the second item of series (3.6), the absolute value of every item of (3.6) is less than corresponding item of positive series.

$$1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}$$

Moreover, the positive series is convergent, further, by Weierstrass criterion, series (3.6) is convergent on $(-\infty, T]$, furthermore, it is uniformly convergent on $(-\infty, T]$. The sum function be $x(t)$, therefore approximate sequence \{x^n(t)\} uniformly converge to $x(t)$ on $(-\infty, T]$. Since \{x^n(t)\} is continuous on $x(t)$ on $(-\infty, T]$ and $f_t$-adapted. On the other hand, (3.5) implies that for each $t$, sequence \{x^n(t)\} is also a Cauchy sequence in $L^2$. Hence, as $n \rightarrow \infty$, $x^n(t)L^2x(t)$ that is $E|x^n(t) - x(t)|^2 \rightarrow 0$. Let $n \rightarrow \infty$,

$$E|x(t)|^2 \leq C_1e^{4L_2(1+2T^3)}, 0 \leq t \leq T.$$

by using the above result,

$$E \int_{-\infty}^{T} |x(s)|^2ds \leq E \int_{-\infty}^{t} |x(s)|^2ds + E \int_{t}^{T} |x(s)|^2ds < \infty.$$

Now show that $x(t)$ satisfies (3.1)

$$E\int_{0}^{t} \{f(s,x_n^s) - f(s,x_s^t)\}ds +$$

$$\int_{0}^{t} \int_{0}^{s} g(s,u,x_u^s)du - \int_{0}^{s} g(s,u,x_u^0)du|ds$$

$$+ \int_{0}^{t} \{\sigma(s,x_n^s) - \sigma(s,x_s^s)\}dB(s)|^2$$

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\[ \leq 3L_1(1 + 2T) \int_0^t E\|x^n_s - x_s\|^2 \, ds \]

\[ M \int_0^t E\|x^n_s - x_s\|^2 \, ds \]

\[ \leq M \int_0^t E\|x^n(s) - x(s)\|^2 \, ds \]

Letting that sequence \( x^n(t) \) is uniformly convergent on \( (-\infty, T] \), it means that \( \epsilon > 0 \), there exists as \( \eta \geq \eta_0 \) for any \( t \in (-\infty, T] \), one then deduces that

\[ E|\|x^n(t) - x(t)\|^2 < \epsilon, \]

\[ \int_0^T E\|x^n(s) - x(s)\|^2 < T\epsilon \]

In other words, for \( t \in [0, T] \)

\[ \int_0^t f(s, x^n_s) \, ds \rightarrow L^2 \int_0^t f(s, x_s) \, ds \]

\[ \int_0^t g(s, u, x^n_u) \, du \rightarrow L^2 \int_0^t g(s, u, x_u) \, du \]

\[ \int_0^t \sigma(s, x^n_s) \, ds \rightarrow L^2 \int_0^t \sigma(s, x_s) \, ds \]
for $0 \leq t \leq T$, taking limits on both sides of
\[
\lim_{n \to \infty} x^n(t) = \xi(0) + \lim_{n \to \infty} \int_0^t f(s, x^{n-1}_s)ds
+ \lim_{n \to \infty} \int_0^t \left[ \int_0^s g(s, u, x^{n-1}_u)du \right]ds
+ \lim_{n \to \infty} \int_0^t \sigma(s, x^{n-1}_s)dB(s).
\]
The above results demonstrate that $x(t)$ is the solution of (3.1).

Next, to prove uniqueness. Let $x(t)$ and $\overline{x(t)}$ be any two solutions of (3.1), by Lemma (3.1), $x(t), \overline{x(t)} \in \mathcal{M}^2((−\infty, T]: \mathbb{R}^d)$.

\[
|x(t) - \overline{x(t)}|^2 = |\int_0^t [f(s, x_s) - f(s, \overline{x_s})]ds
+ \int_0^t \left[ \int_0^s g(s, u, x_u)du - \int_0^s g(s, u, \overline{x_u})du \right]ds
+ \int_0^t \{\sigma(s, x_s) - \sigma(s, \overline{x_s})\}dB(s)|^2
\]

\[
|\int_0^t [f(s, x_s) - f(s, \overline{x_s})]ds|^2 \leq 3|\int_0^t [f(s, x_s) - f(s, \overline{x_s})]ds|^2
+ 3|\int_0^t \{\sigma(s, x_s) - \sigma(s, \overline{x_s})\}dB(s)|^2
\]

\[
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\]
taking expectation on both sides

\[ |x(t) - \overline{x(t)}|^2 \leq 3TE \int_0^t |f(s, x_s) - f(s, \overline{x_s})|^2 ds \]

\[ 3TE \int_0^t \left[ \int g(s, u, x_u) du - \int g(s, u, \overline{x_u}) du \right] ds \]

\[ + 3E \int_0^t \{|\sigma(s, x_s) - \sigma(s, \overline{x_s})|dB(s)|^2 \]

\[ \leq (3L_1T + 3L_1T + 3L_1) \int_0^t E |x_s - \overline{x_s}|^2 ds \]

\[ \leq 3L_1(1 + 2T) \int_0^t E |x_s - \overline{x_s}|^2 ds \]

\[ E(\sup_{0 \leq t \leq T} |x(t) - \overline{x(t)}|^2) \leq 3L_1(1 + 2T) \int_0^t E(\sup_{0 \leq s \leq t} |x(s) - \overline{x(s)}|^2) ds \]

Applying the Gronwall inequality,

\[ E(\sup_{0 \leq s \leq t} |x(s) - \overline{x(s)}|^2) = 0. \]

Thus the above \( x(t) = \overline{x(t)} \) for \( 0 \leq t \leq T \). For all \(-\infty < t \leq T, x(t) = \overline{x(t)} \) a.s, the proof of uniqueness is complete.

The process of the proofs shows that Picard sequence \( x^n(t) \) converges to accurate solution \( x(t) \) of (3.1). It tells us how to get the approximate solution of (3.1) and how to construct Piard sequence \( x^n(t) \).

**Theorem 15.** Suppose that assumptions of theorem (3.2) hold. Let \( x(t) \) be the solution of (3.1) with initial data (3.2) and \( x^n(t) \) is defined by (3.4). Then for all \( n \geq 1 \), it follows that

\[ E(\sup_{0 \leq t \leq T} |x^n(t) - x(t)|^2) \leq \frac{2C[M^2]^n}{n!} e^{2MT} \]

where \( C = 3L_2T(1 + 2T) + 3L_2(1 + 2T)E \|\xi\| ^2, M = 3L_1(1 + 2T). \)
Proof. The discussion is similar to Theorem (3.1), then

\[ E(\sup_{0 \leq t \leq T} |x^n(t) - x(t)|^2) \leq M \int_0^t E\|x^{n-1}_s - x_s\|^2 \, ds \]

\[ \leq M \int_0^t E(\sup_{0 \leq r \leq s} |x^{n-1}(r) - x(r)|^2) \, ds \]

\[ \leq 2M \int_0^t E(\sup_{0 \leq r \leq s} |x^n(r) - x^{n-1}(r)|^2) \, ds \]

\[ + 2M \int_0^t E(\sup_{0 \leq r \leq s} |x^n(r) - x(r)|^2) \, ds \]

Substitute (3.5) into the above expression, then deduce that

\[ E(\sup_{0 \leq t \leq T} |x^n(t) - x(t)|^2) \leq 2C[M^n] (n-1)! + 2M \int_0^t E(\sup_{0 \leq r \leq s} |x^n(r) - x(r)|^2) \, ds \]

Making use of the Gronwall inequality,

\[ E(\sup_{0 \leq t \leq T} |x^n(t) - x(t)|^2) \leq 2C[M^n] (n)! e^{2MT}, 0 \leq t \leq T. \]

The proof is complete.

References


