Common Fixed Point Theorem in
Intuitionistic Generalized Fuzzy Metric
Space Using Implicit Relation

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Abstract

In this paper, our objective is to prove a common fixed point theorem by removing the assumption of continuity, relaxing compatibility to weak compatibility and replacing the completeness of the space with a set of four alternative conditions for functions satisfying an implicit relation in intuitionistic generalized fuzzy metric space.

Key Words: Intuitionistic generalized fuzzy metric spaces, weakly compatible maps, implicit relation, property (E.A.).

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1. **Introduction**

Zadeh [16] introduced the concept of fuzzy sets in 1965 and in the next decade Kramosil and Michalek [8] introduced the concept of fuzzy metric space in 1975, which opened an avenue for further development of analysis in such spaces. Consequently, in due course of time some metric fixed point results were generalized to fuzzy metric space by various authors viz George and Veeramani [5], Grabiec [6], Subrahmanyam [15] and others.

In 1994, Mishra, Sharma and Singh [10] introduced the notion of compatible maps under the name of asymptotically commuting maps in FM-spaces. Singh and Jain [14] studied the notion of weak compatibility in FM-space. (Introduced by Jungck and Rhoades [7] in metric space). However, the study of common fixed points of noncompatible maps is also of great interest. Pant [11] initiated the study of common fixed points of noncompatible maps in metric spaces. In 2002, Aamri and Moutawakil [1] studied a new property for pair of maps i.e. the so-called property (E.A), which is a generalization of the concept of non compatible maps in metric spaces. Recently, Pant and Pant [12] studied the common fixed points of a pair of noncompatible maps and the property (E.A) in fuzzy metric space.

2. **Preliminaries**

**Definition 2.1**

A binary operation \( \ast : [0, 1] \times [0,1] \rightarrow [0,1] \) is a continuous t - norm if it satisfies the following conditions

1. \( \ast \) is associative and commutative,
2. \( \ast \) is continuous,
3. \( a \ast 1 = a \), for all \( a \in [0, 1] \),
4. \( a \ast b \leq c \ast d \) whenever \( a \leq c \) and \( b \leq d \), for each \( a, b, c, d \in [0, 1] \).

Examples for continuous t – norm are \( a \ast b = \min\{a, b\} \) and \( a \ast b = ab \)

**Definition 2.2**

A binary operation \( \diamond : [0, 1] \times [0,1] \rightarrow [0,1] \) is a continuous t - co norm if it satisfies the following conditions

1. \( \diamond \) is associative and commutative,
2. \( \diamond \) is continuous,
3. \( a \diamond 0 = a \), for all \( a \in [0, 1] \),
4. \( a \diamond b \leq c \diamond d \) whenever \( a \leq c \) and \( b \leq d \), for each \( a, b, c, d \in [0, 1] \).

Examples for continuous t – co norm are \( a \diamond b = \max\{a, b\} \) and \( a \diamond b = a + b - ab \)
Note that among a number of possible choices for $\ast$, $a \ast b = \min\{a, b\}$ and $a \bullet b = \max\{a, b\}$ or simply $\ast = \min$and $\bullet = \max$ is the strongest possible universal t-norm and t-conorm.

**Definition 2.3**

A 5-tuple $(X, \mathcal{M}, \mathcal{N}, \ast, \bullet)$ is called an intuitionistic generalized fuzzy metric space if $X$ is an arbitrary (non-empty) set, $\ast$ is a continuous t-norm, $\bullet$ a continuous t-conorm and $\mathcal{M}, \mathcal{N}$ are fuzzy sets on $X^3 \times (0, \infty)$, satisfying the following conditions:

for each $x, y, z, a \in X$ and $t, s > 0$.

a) $\mathcal{M}(x, y, z, t) + \mathcal{N}(x, y, z, t) \leq 1$,

b) $\mathcal{M}(x, y, z, t) > 0$,

c) $\mathcal{M}(x, y, z, t) = 1$ if and only if $x = y = z$,

d) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p(x, y, z), t)$, where $p$ is a permutation function,

e) $\mathcal{M}(x, y, z, a, t) \ast \mathcal{M}(a, z, z, s) \leq M(x, y, z, t + s)$,

f) $\mathcal{M}(x, y, z, .) : (0, \infty) \rightarrow [0, 1]$ is continuous,

g) $\mathcal{N}(x, y, z, t) > 0$,

h) $\mathcal{N}(x, y, z, t) = 0$, if and only if $x = y = z$,

i) $\mathcal{N}(x, y, z, t) = \mathcal{N}(p(x, y, z), t)$ where $p$ is a permutation function,

j) $\mathcal{N}(x, y, z, a, t) \bullet \mathcal{N}(a, z, z, s) \geq \mathcal{N}(x, y, z, t + s)$,

k) $\mathcal{N}(x, y, z, .) : (0, \infty) \rightarrow [0, 1]$ is continuous.

Then $(\mathcal{M}, \mathcal{N})$ is called an intuitionistic generalized fuzzy metric on $X$.

In the following example, we know that every fuzzy metric induces a intuitionistic generalized fuzzy metric.

**Example 2.4**

Let $(X, d)$ be a metric space. Define $a \ast b = a.b$ (or $a \ast b = \min\{a, b\}$) for all $x, y, z \in X$ and $t > 0$, $\mathcal{M}(x, y, z, t) = \frac{d(x, y, z)}{t + d(x, y, z)}$ and $\mathcal{N}(x, y, z, t) = \frac{d(x, y, z)}{t + d(x, y, z)}$.

Then $(X, \mathcal{M}, \mathcal{N}, \ast, \bullet)$ is a intuitionistic generalized fuzzy metric space and the fuzzy metric $\mathcal{M}, \mathcal{N}$ induced by the metric $D$ is often referred to as the standard fuzzy metric.

**Definition 2.5**

Let $A$ and $B$ maps from a intuitionistic generalized fuzzy metric space $(X, \mathcal{M}, \mathcal{N}, \ast, \bullet)$ into itself. The maps $A$ and $B$ are said to be compatible
(or asymptotically commuting), if for all t, $\lim_{n \to \infty} M(AB_{x_n}, BA_{x_n}, BA_{x_n}, t) = 1$ and $\lim_{n \to \infty} N(AB_{x_n}, BA_{x_n}, BA_{x_n}, t) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} A_{x_n} = \lim_{n \to \infty} B_{x_n} = 1$ for some $z \in X$.

From the above definition it is inferred that A and B are non compatible maps from a IGFM-space $(X, M, \cdot, *; N, \cdot, *)$ into itself if $\lim_{n \to \infty} A_{x_n} = \lim_{n \to \infty} B_{x_n} = z$ for some $z \in X$, but either $\lim_{n \to \infty} M(AB_{x_n}, BA_{x_n}, BA_{x_n}, t) \neq 1$ and $\lim_{n \to \infty} N(AB_{x_n}, BA_{x_n}, BA_{x_n}, t) \neq 0$ or the limit does not exist.

**Definition 2.6**

Let A and B maps from an intuitionistic generalized fuzzy metric space $(X, M, \cdot, *; N, \cdot, *)$ into itself. The maps A and B are said to be weakly compatible if they commute at their coincidence points, that is, $A_{x_n} = B_{x_n}$ implies that $A_{x_n} = BA_{x_n}$. Note that compatible mappings are weakly compatible but converse is not true in general.

**Definition 2.7**

Let A and B be two self-maps of a IGFM-space $(X, M, \cdot, *; N, \cdot, *)$. We say that A and B satisfy the property (E. A) if there exists a sequence $\{x_n\}$ such that $\lim_{n \to \infty} A_{x_n} = \lim_{n \to \infty} B_{x_n} = z$ for some $z \in X$.

Note that weakly compatible and property (E.A) are independent to each other.

### 3. Implicit Relation

In our result, we deal with the implicit relation used in [2]. In [2], Altın and Turkoglu used the t-conorm and F, G be the set of all real continuous functions $F, G: I^6 \to \mathbb{R}$ satisfying the following conditions:

- (F-1) $F$ is non increasing in the fifth and sixth variables and $G$ is non decreasing in the fifth and sixth variables,

- (F-2) if, for some constant $k \in (0,1)$ we have
  
  - (F-a) $F(u(kt), v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) \cdot v\left(\frac{t}{2}\right)) \geq 1$ or
  
  - (F-b) $F(u(kt), v(t), u(t), v(t), u\left(\frac{t}{2}\right) \cdot v\left(\frac{t}{2}\right), 1) \geq 1$ and
  
  - (F-c) $G(u(kt), v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) \cdot v\left(\frac{t}{2}\right)) \leq 1$ or
For any fixed $t > 0$ and nondecreasing function $u, v : (0, \infty) \to I$ with $0 \leq u(t), v(t) \leq 1$ then there exists $h \in (0,1)$ with $u(ht) \geq v(t) \ast u(t)$ and $u(ht) \leq (1 - v(t)) \circ (1 - u(t))$.

(F-3) if, for some constant $k \in (0,1)$ we have $F \left( u(kt), u(t), 1, 1, u(t), u(t) \right) \geq 1$ and $G \left( u(kt), u(t), 1, 1, u(t), u(t) \right) \leq 1$.

For any fixed $t > 0$ and nondecreasing function $u : (0, \infty) \to I$ then $u(kt) \geq u(t)$.

4. Main Results

Theorem 4.1

Let $(X, \mathcal{M}, N, \ast, \Phi)$ be a complete intuitionistic generalized fuzzy metric space with $a \ast b = \min \{a, b\}$ and $a \Phi b = \min \{1, a + b\}$ for all $a, b \in I$ and $A, B, S$ and $T$ be maps from $X$ into itself satisfying the conditions:

(4.1.1) $A(X) \subseteq S(T(X), B(X) \subseteq SS(X)$,

(4.1.2) One of the maps $A, B, S$ and $T$ is continuous,

(4.1.3) $(A,S)$ and $(B,T)$ are compatible of type $(\alpha)$,

(4.1.4) There exist $k \in (0,1)$ and $F \in F$ and $G \in G$ such that

$F \left( \mathcal{M}(A, B, By, By, kt), \mathcal{M}(Sx, Ty, Ty, t), \mathcal{M}(Ax, Sx, Sx, t) \right) \geq 1$ and

$G \left( \mathcal{N}(A, B, By, By, kt), \mathcal{N}(x, Ty, Ty, t), \mathcal{N}(Ax, Sx, Sx, t) \right) \leq 1$ for all $x, y \in X, t \geq 0$.

Then $A, B, S$ and $T$ have a unique common fixed point in $X$.

Now, we prove the following

Theorem 4.2

Let $(X, \mathcal{M}, N, \ast, \Phi)$ be a intuitionistic generalized fuzzy metric space with $a \ast b = \min \{a, b\}$ and $a \Phi b = \min \{1, a + b\}$ for all $a, b \in I$. Further, let $(A, S)$ and $(B, T)$ be weakly compatible pairs of self - maps of $X$ satisfying (4.1.1), (4.1.2) and (4.1.4), $(A, S)$ or $(B, T)$ satisfies the property (E.A). If the range of one of the maps $A, B, S$ or $T$ is a
complete sub space of X then A, B, S and T have a unique common fixed point in X.

**Proof:**

If the pair (B, T) satisfies the property (E.A), then there exists a sequence \( \{x_n\} \) in X such that \( Bx_n \to z \) and \( Tx_n \to z \), for some \( z \in X \) as \( n \to \infty \).

Since \( B(X) \subseteq S(X) \) there exists a sequence \( \{y_n\} \) in X such that \( Bx_n = Ty_n \).

Hence, \( Sy_n \to z \) as \( n \to \infty \). Also, since \( A(X) \subseteq S(X) \), there exists a sequence \( \{y'_n\} \) in X such that \( Ay'_n = Tx_n \). Hence, \( Ay'_n \to z \) as \( n \to \infty \).

Suppose that \( S(X) \) is a complete subspace of \( X \). Then, \( z = Su \) for some \( u \in X \). Subsequently, we have \( Ay'_n \to Su, Bx_n \to Su, Tx_n \to Su \) and \( Sy_n \to su \) as \( n \to \infty \).

By (4.1.4), we have

\[
F(Au, Bx_n, Bx_n, k t), M(Sx_n, Tx_n, Tx_n, t), M(Au, Su, Su, t)
\]

\[
M(Bx_n, Tx_n, Tx_n, t), M(Au, Tx_n, Tx_n, t), M(Bx_n, Su, Su, t) \geq 1
\]

\[
G(N(Au, Bx_n, Bx_n, k t), N(Sx_n, Tx_n, Tx_n, t), N(Au, Su, Su, t)),
\]

\[
N(Bx_n, Tx_n, Tx_n, t), N(Au, Tx_n, Tx_n, t), N(Bx_n, Su, Su, t) \leq 1.
\]

Letting \( n \to \infty \), we have

\[
F(M(Au, Su, Su, kt), 1, M(Au, Su, Su, t), 1, M(Au, Su, Su, t), 1) \geq 1
\]

\[
G(N(Au, Su, Su, kt), 0, N(Au, Su, Su, t), 0, N(Au, Su, Su, t), 0) \leq 1.
\]

On the other hand, since

\[
M(Au, Su, Su, t) \geq M\left(Au, Su, Su, \frac{t}{2}\right) = M\left(Au, Su, Su, \frac{1}{2}\right) * 1
\]

\[
N(Au, Su, Su, t) \leq N\left(Au, Su, Su, \frac{t}{2}\right) = N\left(Au, Su, Su, \frac{1}{2}\right) \triangleq 0.
\]

F is nonincreasing and G is non decreasing in the fifth variable, we have, for any \( t > 0 \)

\[
F\left(M(Au, Su, Su, kt), 1, M(Au, Su, Su, t), 1, M(Au, Su, Su, t), 1\right) \geq F(M(Au, Su, Su, kt), 1, M(Au, Su, Su, t), 1, M(Au, Su, Su, t), 1) \geq 1
\]

\[
G(N(Au, Su, Su, kt), 0, N(Au, Su, Su, t), 0, N(Au, Su, Su, t), 0) \geq 1.
\]
which implies, by (F-2) that \( Au = Su \). The weak compatibility of \( A \) and \( S \) implies that \( ASu = SAu \) and then \( = ASu = SSu \).

On the other hand, since \( A(X) \subseteq T(X) \), there exists \( a \in X \) such that \( Au = Tv \).

We now show that \( T = Bv \). By (4.1.4), we have
\[
\begin{align*}
F \left( M(Au, Bu, Bv, kt), M(Su, Tv, Tv, t), M(Au, Su, Su, t) \right) & \geq 1 \quad \text{and} \\
G \left( N(Au, Bu, Bv, kt), N(Su, Tv, Tv, t), N(Au, Su, Su, t) \right) & \leq 1.
\end{align*}
\]

That is,
\[
\begin{align*}
F(M(Tv, Bv, Bv, kt), 1, 1, M(Bv, Tv, Tv, t), 1, M(Bv, Tv, Tv, t)) & \geq 1 \quad \text{and} \\
G(N(Tv, Bv, Bv, kt), 1, 1, N(Bv, Tv, Tv, t), 1, N(Bv, Tv, Tv, t)) & \leq 1.
\end{align*}
\]

On the other hand, since
\[
\begin{align*}
M(Bv, Tv, Tv, t) & \geq M(Bv, Tv, Tv, 1) = M(Bv, Tv, Tv, 1) * 1 \quad \text{and} \\
N(Bv, Tv, Tv, t) & \leq N(Bv, Tv, Tv, 1) = N(Bv, Tv, Tv, 1) \cdot 0,
\end{align*}
\]

and \( F \) is non-increasing and \( G \) is non-decreasing in the sixth variable, we have, for any \( t > 0 \)
\[
\begin{align*}
F \left( M(Bv, Tv, Tv, kt), 1, 1, M(Bv, Tv, Tv, t), 1, M(Bv, Tv, Tv, t) \right) & \geq 1 \quad \text{and} \\
G(N(Bv, Tv, Tv, kt), 0, 0, N(Bv, Tv, Tv, t), 0, N(Bv, Tv, Tv, t)) & \leq 1,
\end{align*}
\]

which implies, by (F-2), that \( Bv = Tp \). This implies that \( Au = Su = Tp = Bv \). The weak compatibility of \( B \) and \( T \) implies that \( BTv = TBv \) and then \( TTv = TBv = BTv = BBv \).
Let us show that $Au$ is a common fixed point of $A$, $B$, $S$ and $T$.

In view of (4.1.4), it follows

$$F\left(\frac{M(AAu, Bv, Bv, kt)}{M(Bv, Tv, t)}, \frac{M(SAu, Tv, Tv, t), M(AAu, SaAu, SaAu, t)}{M(AAu, SaAu, SaAu, t)}\right) \geq 1$$

and

$$G\left(\frac{N(AAu, Bv, Bv, kt), N(SAu, Tv, Tv, t), N(AAu, SaAu, SaAu, t)}{N(Bv, Tv, t), N(AAu, Tv, Tv, t), N(Bv, SaAu, SaAu, t)}\right) \leq 1.$$  

That is,

$$F\left(\frac{M(AAu, Au, Au, kt)}{M(Au, AAu, AAu, t)}\right) \geq 1$$

and

$$G\left(\frac{N(AAu, Au, Au, kt)}{N(AAu, Au, Au, t)}\right) \leq 1.$$  

Thus, from (F-3), we have $M(AAu, Au, Au, kt) \geq M(AAu, Au, Au, t)$ and $N(AAu, Au, Au, kt) \leq N(AAu, Au, Au, t)$, we have, $AAu = Au$.

Therefore, $Au = AAu = SaAu$ and $Au$ is a common fixed point of $A$ and $S$.

Similar, we can prove that $Bv$ is common fixed point of $B$ and $T$. Since $Au = Bv$ we conclude that $Au$ is a common fixed point of $A, B, S$ and $T$. The proof is similar when $T(X)$ is assumed to be a complete subspace of $X$. The cases in which $A(X)$ or $B(X)$ is a complete subspace of $X$ are similar to the cases in which $T(X)$ or $S(X)$ respectively, is complete since $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.

If $Au = Bu = Tu = Su = u$ and $Av = Bv = Tv = Sv = v$, then (4.1.4) gives

$$F\left(\frac{M(Au, Bv, Bv, kt)}{M(Bv, Tv, t)}, \frac{M(Su, Tv, Tv, t), M(Au, Su, Su, t)}{M(Au, Tv, Tv, t), M(Bv, Su, Su, t)}\right) \geq 1$$

and

$$G\left(\frac{N(Au, Bv, Bv, kt)}{N(Su, Tv, Tv, t)}, \frac{N(Au, Su, Su, t)}{N(Bv, Tv, t), N(Au, Tv, Tv, t), N(Bv, Su, Su, t)}\right) \leq 1.$$  

That is,

$$F\left(\frac{M(u, v, v, kt)}{M(u, v, v, t)}, \frac{M(u, v, v, t)}{M(u, v, v, t)}\right) \geq 1$$

and

$$F\left(\frac{M(u, v, v, t)}{M(v, u, t)}\right) \geq 1.$$
Thus, from (F-3), we have \( N(u, v, v, t) \leq N \) and
\( N(u, v, t) \leq N(u, v, v, t) \). We have \( u = v \).
Therefore, \( u = v \) and the common fixed point is unique.

References


