

CHANGING AND UNCHANGING OF ECCENTRIC DOMINATION NUMBER IN GRAPHS

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Abstract

A subset D of the vertex set $V(G)$ of a graph G is said to be a dominating set if every vertex not in D is adjacent to atleast one vertex in D . A dominating set D is said to be an eccentric dominating set if for every $v \in V - D$, there exists atleast one eccentric point of v in D . The minimum of the cardinalities of the eccentric dominating sets of G is called the eccentric domination number $\gamma_{ed}(G)$ of G . In this paper, we have studied the changing and unchanging of eccentric domination number in graphs.

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1 Introduction

Let G be a finite, simple, undirected graph on n vertices with vertex set $V(G)$ and edge set $E(G)$. For graph theoretic terminology refer Harary [5], Buckley and Harary [1].

The concept of distance in graphs plays a dominant role in the study of structural properties of a graph in various angles using related concept of eccentricity of vertices in graphs. The study of structural properties of graphs using distance and eccentricity started with the study of center of tree and propagated in different directions in the study of structural properties of graph such as unique eccentric point graphs. K -eccentric point graphs, self centered graphs.

Let G be a connected graph and u be a vertex of G . The eccentricity $e(v)$ of v is the distance to a vertex farthest from v . Thus $e(v) = \max\{d(u, v) : u \in V\}$. The radius $r(G)$ is the minimum eccentricity of the vertices, whereas the diameter $diam(G)$ is the maximum eccentricity. For any connected graph G , $r(G) \leq diam(G) \leq 2r(G)$. The vertex v is a central vertex of G if $e(v) = r(G)$. The center $C(G)$ is the set of all central vertices of G . The central subgraph $\langle C(G) \rangle$ of a graph G is the subgraph induced by the center. The vertex v is a peripheral vertex if $e(v) = diam(G)$. The periphery $P(G)$ is the set of all peripheral vertices of G .

For a vertex v , each vertex at a distance $e(v)$ from v is an eccentric vertex of v . Eccentric set of a vertex v is defined as $E(v) = \{u \in V(G) : d(u, v) = e(v)\}$. The open neighborhood $N(u)$ of a vertex u is the set of all vertices adjacent to u in V . $N[u] = N(u) \cup \{u\}$ is called the closed neighborhood of u . For a vertex $u \in V(G)$, $N_i(u) = \{u \in V(G) : d(u, v) = i\}$ is defined to be the i^{th} neighborhood of u in G .

The concept of domination in graphs was introduced by Ore [8] and Cockayne et al. studied various bounds and results to domination in [4].

A set $D \subseteq V$ is said to be a dominating set of G , if every vertex in $V - D$ is adjacent to some vertex in D .

In 2010, Janakiraman, Bhanumathi and Muthammai defined eccentric domination in graphs [6] and studied eccentric domination in trees [2] and various bounds of eccentric domination in graphs

[3].

The changing and unchanging terminology was first suggested by Harary [5]. It is useful to partition the vertex set or the edge set of a graph G into three sets according to how the addition of vertex (or edge) or removal of addition (edge) affects the domination number.

A set $D \subseteq V(G)$ is an eccentric dominating set if D is a dominating set of G and for every $v \in V - D$, there exists atleast one eccentric vertex of v in D . If D is an eccentric dominating set, then every superset $D' \supseteq D$ is also an eccentric dominating set. But $D'' \subseteq D$ is not necessarily an eccentric dominating set. An eccentric dominating set D is a minimal eccentric dominating set if no proper subset $D'' \subseteq D$ is an eccentric dominating set.

The eccentric domination number $\gamma_{ed}(G)$ of a graph G is the minimum cardinality of an eccentric dominating set.

In this paper, we have studied the changing and unchanging of the above eccentric domination number in graphs.

2 Prior Results

Theorem 1. 1. $\gamma_{ed}(K_n) = 1, n \geq 3$.

2. $\gamma_{ed}(K_{1,n}) = 2, n \geq 2$.

3. $\gamma_{ed}(W_n) = 3, n \geq 7$.

4. $\gamma_{ed}(K_{m,n}) = 2, n \geq 2$.

5. $\gamma_{ed}(P_n) = \begin{cases} \lceil n/3 \rceil & \text{if } n = 3k + 1 \\ \lceil n/3 \rceil + 1 & \text{if } n = 3k \text{ or } 3k + 2. \end{cases}$

6. $\gamma_{ed}(C_n) = \begin{cases} n/3 & \text{if } n = 3m \text{ and is odd} \\ \lceil n/3 \rceil & \text{if } n = 3m + 1 \text{ and is odd} \\ \lceil n/3 \rceil + 1 & \text{if } n = 3m + 2 \text{ and is odd.} \end{cases}$

Theorem 2. Let n be an even integer. Let G be obtained from the complete graph K_n by deleting edges of a linear factor. Then $\gamma_{ed}(G) = \frac{n}{2}$.

Theorem 3. *Let G be a connected graph. Then $\gamma_{ed}(G) = 2$ if and only if G is any one of the following.*

- (i) $r(G) = 1$, $d(G) = 2$ and $u \in V(G)$ such that $e(u) = 2$ and $d(u, v) = 2$ for all $v \in V(G)$ with $e(v) = 2$.
- (ii) G is self centered of diameter 2, having a dominating edge which is not in a triangle.
- (iii) $r(G) = 2$, $d(G) = 3$ and G has a γ -set D of cardinality two which is not connected.

3 Changing and Unchanging of $\gamma_{ed}(G)$ Vertex Removal

If we remove a vertex from a graph G , then the vertex set of G is partitioned into three sets according to how its removal changes, unchanges the eccentric domination number. Also removal of a vertex can increase or decrease the eccentric domination number. In some graphs, there are vertices whose removal will maintain (unchanged) the eccentric domination number. Thus, we have the following 3 sets.

$$\begin{aligned} VD^0 &= \{v \in V(G) / \gamma_{ed}(G - v) = \gamma_{ed}(G)\} \\ VD^- &= \{v \in V(G) / \gamma_{ed}(G - v) < \gamma_{ed}(G)\} \\ VD^+ &= \{v \in V(G) / \gamma_{ed}(G - v) > \gamma_{ed}(G)\} \end{aligned}$$

Then $V = VD^0 \cup VD^- \cup VD^+$ and the following are the results of some special classes of graphs.

Theorem 4.

1. If G is a complete graph with atleast three vertices then $V = VD^0$.
2. Let G be a star $K_{1,n}$, $n \geq 3$, (i) If v is a pendant vertex then $v \in VD^0$ (ii) If v is a central vertex then $v \in VD^+$.
3. If G is a complete bipartite graph then $V = VD^0$ for $m, n \geq 3$.

4. If G is a path on atleast four vertices and v is a pendant vertex of G ,
 $v \in VD^-$, for $n = 3k + 1$ and
 $v \in VD^0$, for $n = 3k$ or $3k + 1$.
5. If G is a wheel and v is not a central vertex of G then $v \in VD^0$ for $n \geq 3$.
 If G is a wheel and v is a central vertex of the wheel then $v \in VD^+$ for $n \geq 11$. Here $W_n = C_n + K_1$.
6. Let G be a cycle C_n . If n is even then $v \in VD^-$. If n is odd then (i) $V = VD^+$ for $n = 3m$ (ii) $V = VD^0$ for $n = 3m + 1$ (iii) $V = VD^-$ for $n = 3m + 2$.
7. If G is a corona $C_n \circ K_1$ ($n \geq 3$), and if v is a pendant vertex of G , then $v \in VD^0$.

Proof.

- (i) $G \cong K_n$, $n \geq 3$ by Theorem 1 $\gamma_{ed}(K_n) = 1$. Let $v \in V(G)$. Then $G - v \simeq K_{n-1}$ and $\gamma_{rd}(G - v) = 1 = \gamma_{ed}(G)$. Therefore $v \in VD^0$ and hence $V = VD^0$.
- (ii) Let $G \cong K_{1,n}$, for $n > 3$. By Theorem 1 $\gamma_{ed}(K_{1,n}) = 2$. Let $v \in V(G)$ be the central vertex then $G - v = nK_1$, totally disconnected. Therefore $\gamma_{ed}(G - v) = \gamma(G - v) = n$. Thus $v \in VD^+$. Let $v \in V(G)$ be a pendant vertex. Then $G - v \simeq K_{1,n-1}$ and $\gamma_{ed}(G - v) = 2 = \gamma_{ed}(G)$. Therefore $v \in VD^0$.
- (iii) Let G be a complete bipartite graph $K_{m,n}$, where $m, n \geq 2$. By Theorem 1 $\gamma_{ed}(K_{m,n}) = 2$. Let $v \in V(G)$. Then $\gamma_{ed}(G - v) = 2 = \gamma_{ed}(G)$. Therefore $v \in VD^0$ and hence $V = VD^0$.
- (iv) Let $G \cong P_n$, $n \geq 4$ by Theorem 1.

$$\gamma_{ed}(P_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil, & \text{if } n = 3k + 1 \\ \left\lceil \frac{n}{3} \right\rceil + 1, & \text{if } n = 3k \text{ or } 3k + 2 \end{cases}$$

Let v be a pendant vertex in P_n . Then $G - v \simeq P_{n-1}$, $\gamma_{ed}(G - v) < \gamma_{ed}(G)$ for $n = 3k + 2$. Therefore $v \in VD^-$.

$\gamma_{ed}(G - v) = \gamma_{ed}(G)$ for $n = 3k$ or $3k + 1$, therefore $v \in VD^0$.

- (v) Let G be a wheel W_n on $(n+1)$ vertices, where $W_n = C_n + K_1$, $n \geq 11$. Then by Theorem 1 $\gamma_{ed}(W_n) = 3$. Let v be a vertex of W_n .

Case (i): Let $v \in V(C_n)$. Then $G - v \simeq K_1 + P_{n-1}$ and $\gamma_{ed}(G - v) = 3 = \gamma_{ed}(G)$. Hence $v \in VD^0$.

Case (ii): Let v be the central vertex of W_n . Then, $G - v \simeq C_n$ and $\gamma_{ed}(G - v) \geq \gamma_{ed}(G)$. Hence $v \in VD^+$.

- (vi) Let $G \cong C_n$. By Theorem 1.

$\gamma_{ed}(C_n) = \frac{n}{2}$ if n is even.

$$\gamma_{ed}(C_n) = \begin{cases} \frac{n}{3}, & \text{if } n = 3m \text{ and is odd} \\ \left\lceil \frac{n}{3} \right\rceil, & \text{if } n = 3m + 1 \text{ and is odd.} \\ \left\lceil \frac{n}{3} + 1 \right\rceil, & \text{if } n = 3m + 2 \text{ and is odd} \end{cases}$$

$G - v \cong P_{n-1}$.

Case (i): If n is even, then $\gamma_{ed}(G - v) < \gamma_{ed}(G)$. Therefore $v \in VD^-$, $V = VD^-$.

Case (ii): If n is odd, then

$\gamma_{ed}(G - v) > \gamma_{ed}(G)$ for $n = 3m$, Therefore $v \in VD^-$, $V = VD^+$.

$\gamma_{ed}(G - v) = \gamma_{ed}(G)$ for $n = 3m + 1$. Therefore $v \in VD^0$, $V = VD^0$.

$\gamma_{ed}(G - v) < \gamma_{ed}(G)$ for $n = 3m + 2$. Therefore $v \in VD^-$, $V = VD^-$.

- (vii) Let G be the corona $C_n \circ K_1$ and let v be a pendant vertex of G . Then, $G - v$ is a graph obtained by attaching exactly one pendant edge at each of $(n - 1)$ vertices of C_n . Then a minimum eccentric dominating set of $G - v$ contains all the $(n - 1)$ pendant vertices and a vertex of C_n , and hence $\gamma_{ed}(G - v) = n = \gamma_{ed}(G)$. Therefore $v \in VD^0$.

□

Theorem 5. Let G be a self centered graph of diameter 2, having a dominating edge which is not in a triangle then, $V = VD^0 \cup VD^+$

Proof. Let G be a self centered graph of diameter 2, having a dominating edge which is not in a triangle. Then $\gamma_{ed}(G) = 2$. If we remove any vertex v , then $\gamma_{ed}(G - v) \geq 2$, since $\gamma_{ed}(G - v) \neq 1$. Hence $\gamma_{ed}(G - v) \geq 2 = \gamma_{ed}(G)$ for every $v \in V(G)$ and hence $V = VD^0 \cup VD^+$. $\square$

Theorem 6. If $G = \overline{K}_m + K_1 + K_1 + K_1$, $m > 2$ and v is a pendant vertex then $v \in VD^-$ or VD^0 .

Proof. G is a wounded spider having one nonwounded leg. Also, $\gamma_{ed}(G) = 3 = n - \Delta(T) + 1$

Case (i): When we remove a pendant vertex v adjacent to a central vertex of degree 2, then $\gamma_{ed}(G - v) = 2$. Therefore $v \in VD^-$.

Case (ii): When we remove a vertex v , which is adjacent to a central vertex of degree ≥ 3 , then $\gamma_{ed}(G - v) = 3$. Therefore $v \in VD^0$. $\square$

Theorem 7. If $G = \overline{K}_m + K_1 + K_1 + \overline{K}_n$, $m, n \geq 3$, and v is a pendant vertex then $v \in VD^0$.

Proof. G is a bicentral tree with degree of the central vertices are ≥ 3 . If we remove a pendant vertex from G , then $\gamma_{ed}(G - v) = 4 = \gamma_{ed}(G)$. Therefore $v \in VD^0$. $\square$

Theorem 8. Let G be a connected graph with $|V(G)| = n$, $\gamma_{ed}(G \circ K_1) = n$. If v is a pendant vertex, then $v \in VD^0$.

Proof. Let G be a connected graph with $|V(G)| = n$. Then $\gamma_{ed}(G \circ K_1) = n$. If we remove any pendant vertex then it will not affect the eccentric domination number of G . Therefore $v \in VD^0$. $\square$

Theorem 9. If G is a caterpillar or order n such that each non pendant vertex is of degree three and v is a pendant vertex then $v \in VD^-$.

Proof. Let G be a caterpillar such that each non pendant vertex is of degree three. Therefore $\gamma_{ed}(G) = \frac{n}{2} + 1$.

Case (i): When we remove a pendant vertex v , which is peripheral then $\gamma_{ed}(G - v) = \gamma_{ed}(G) - 1$. Therefore $v \in VD^-$.

Case (ii): When we remove a pendant vertex which is not

peripheral then $\gamma_{ed}(G - v) < \gamma_{ed}(G)$. Therefore $v \in VD^-$.

From Case (i) and (ii) $v \in VD^-$. $\square$

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