

Graph Distance of Vertices in Double Vertex Graphs

Beaula C¹, Venugopal P², Padmapriya N³

^{1,2,3} Department of Mathematics

SSN College of Engineering

Kalavakkam 603 110

¹*beaulac@ssn.edu.in*

²*venugopalp@ssn.edu.in*

³*padmapriyan@ssn.edu.in*

Abstract

The distance between two vertices u and v is the number of edges in a shortest path from u to v , which is denoted by $d(u, v)$. The double vertex graph $U_2(G)$ of a graph G is a graph constructed from G . This paper proposes the distance of vertices of double vertex graph of path, cycle, and wheel.

Keywords: distance of vertices, double vertex graphs, path, cycle, wheel.

AMS Subject Classification: 05C12

1 Introduction

Let $G = (V, E)$ be a finite simple graph, with a non-empty set of vertices V and a set of edges E . The *distance* [4] between two vertices u and v in G is the number of edges in a shortest path from u to v , and is denoted by $d(u, v)$. This is also known as the *geodesic distance*. There may be more than one shortest path between two vertices. If there is no path connecting the two vertices then the value for $d(u, v)$ is taken as infinite. Although this concept has

been known for a very long time, it is only in recent decades that distance has received considerable attention as a subject of its own. In 1990 Buckley and Harary [1] wrote the book *Distance in Graphs*. *The Handbook of Graph Theory*, edited by Gross and Yellen [5], contains a section devoted exclusively to distance in graphs. A number of results on distance come from the fact that two vertices u and v are adjacent if and only if $d(u, v) = 1$ and two distinct vertices u and v are non-adjacent if and only if $d(u, v) \geq 2$. Hence any concept whose definition relies on the adjacency or non-adjacency of two vertices in a graph can be restated in terms of distance, thereby giving rise to a natural generalization of the concept [2]. The distance between two vertices is the basis of the definition of several graph parameters including diameter, radius, average distance and metric dimension [2].

In this paper, the distance between any two vertices of double vertex graphs of path, cycle and wheel, have been studied.

2 Double Vertex Graphs

There are many graph functions with which one can construct a new graph from a given graph or set of graphs, such as Cartesian product and the line graph. One such graph function is called the *double vertex graph*. This was introduced by Alavi et al. in [9] and studied in [8, 6] inter alia. For a survey, see [7]. Let $G = (V, E)$

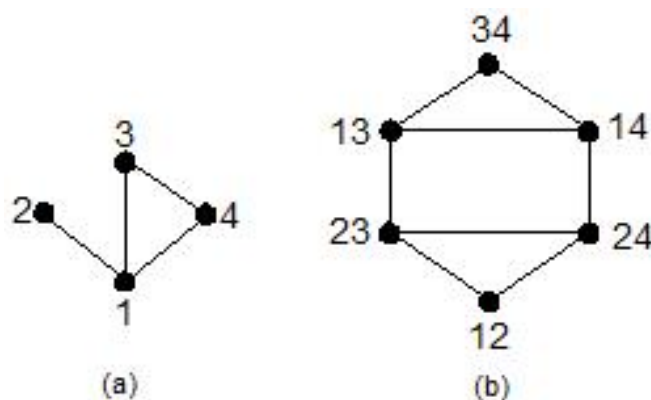


Figure 1: (a) Graph G (b) Double vertex graph of G

be a graph of order $n \geq 2$. The *double vertex graph* $U_2(G)$ is the graph whose vertex set consists of all $n(n-1)/2$ unordered pairs from V such that two vertices $\{x, y\}$ and $\{u, v\}$ are adjacent if and only if $|\{x, y\} \cap \{u, v\}| = 1$ and if $x = u$ then y and v are adjacent in G . The order and size of $U_2(G)$ are $n(n-1)/2$ and $m(n-2)$ respectively, where n is the order and m is the size of G . See Figure 1 for an example of a graph and its double vertex graph.

In this paper, for convenience, the vertices of the double vertex graph of path, cycle and wheel are labeled as ij , $i < j$. In our calculation of distance between two vertices ij and kl we always consider $i \leq k$ and $j \leq l$. $d(ij, kj)$ is the length of the path from ij to kj along j th column and is given by $|k - i|$, $d(ij, il)$ is the length of the path from ij to il along the i th row and is given by $|l - j|$.

3 Double Vertex Graph of a path

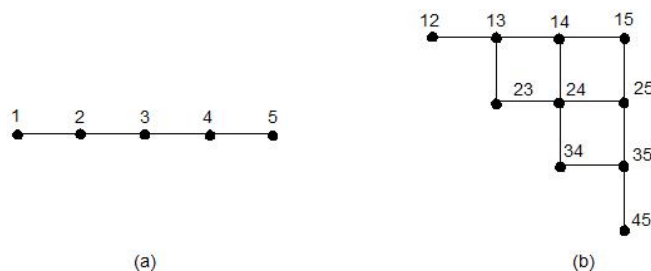


Figure 2: (a) Path P_5 (b) Double vertex graph of path $U_2(P_5)$

In mathematical field of graph theory, a *path graph* or a *linear graph* is a graph whose vertices can be listed in the order $1, 2, 3, \dots, n$ such that the edges are $(i, i+1)$ where $i = 1, 2, 3, \dots, n-1$. Path with n vertices is denoted as P_n . The double vertex graph of a path is denoted as $U_2(P_n)$. In Figure 2, path P_5 and its double vertex graph $U_2(P_5)$ is shown.

The number of vertices in $U_2(P_n)$ is $(n(n-1))/2$, the size of $U_2(P_n)$ is $n^2 - 3n + 2$, since P_n is a tree.

Theorem 3.1 The distance between any two vertices ij and kl in a double vertex graph $U_2(P_n)$ is $d(ij, kl) = |i - k| + |j - l|$.

Proof : Let $u = ij$ and $v = kl$ be any two vertices in $U_2(P_n)$.

Case (1): Let the two vertices u and v be in the same row, i.e., $i = k$ of $U_2(P_n)$.

In this case, the distance between u and v is given by the number of edges between them in that row. Then $d(ij, kl) = |j - l|$

Case (2): Let u and v be the two vertices in different rows, i.e., $i \neq k$ of $U_2(P_n)$. Now these two vertices can be in the same column or in different column.

Case (i): Suppose u and v are in the same column, i.e., $j = l$

$$d(ij, kl) = |i - k| \quad (1)$$

Case (ii): Suppose u and v are in different columns, i.e., $j \neq l$. The distance between ij and kl is found out by traversing along the horizontal path from ij to il and then along the vertical path from il to kl . That is

$$d(ij, kl) = d(ij, il) + d(il, kl) \quad (2)$$

$$d(ij, kl) = |i - k| + |j - l| \quad (3)$$

Thus, for any two vertices ij and kl in a double vertex graph $U_2(P_n)$ the distance between them is

$$d(ij, kl) = |i - k| + |j - l| \quad (4)$$

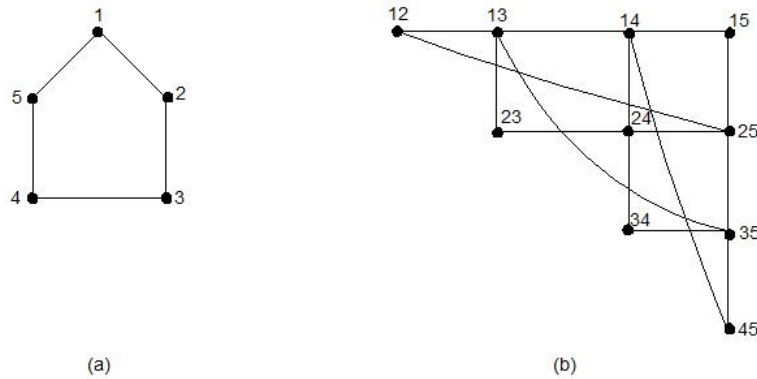
Hence the proof.

4 Double Vertex Graph of a cycle

A *cycle* or a *circular graph* is a closed path whose initial and terminal vertices coincide. The cycle with n vertices is denoted as C_n . The number of vertices and the number of edges are equal in C_n , and every vertex has degree 2. The double vertex graph of a cycle C_n is denoted as $U_2(C_n)$. In Figure 3, cycle C_5 and its double vertex graph $U_2(C_5)$ is shown.

The number of vertices in $U_2(C_n)$ is $(n(n-1))/2$, the size of $U_2(C_n)$ is $m(n-2)$, where m is the size of C_n .

Theorem 4.1 The maximum distance between any two vertices u and v in $U_2(C_n)$ is $d(u, v) \leq (n-2)$

Figure 3: (a) Cycle C_5 (b) Double vertex graph of cycle $U_2(C_5)$

Proof : Let $u = ij$ and $v = kl$ be any two vertices in $U_2(C_n)$ where $i = 1, 2, \dots, (n-1)$ and $j = 2, 3, \dots, n$. For vertex ij , i denotes the row position and j denotes the column position. The distance between any two vertices u and v is calculated as in the following cases.

Case (i): Suppose the shortest path between ij and kl is obtained by traversing along k th row and j th column, then

$$d(ij, kl) = |k - i| + |l - j| \quad (5)$$

Case (ii): Suppose the shortest path between ij and kl exists by traversing, from ij to $1j$ along j th column and through an edge $(1j, jn)$, from jn to kn along k th column, then kn to kl along k th row.

In this case,

$$d(ij, kl) = |i - 1| + 1 + |j - k| + |n - l| \quad (6)$$

Hence by the above cases, the distance between any two vertices ij and kl is obtained by

$$d(ij, kl) = \min \{ |k - i| + |l - j|, [|i - 1| + 1 + |j - k| + |n - l|] \} \quad (7)$$

To prove: $d(u, v) \leq (n - 2)$

As the first vertex 12 is in the first row and $(n - 1)n$ is the last vertex in the last row,

$$\begin{aligned}
d(ij, kl) &\leq d(12, (n-1)n) \\
&= \min \{ [| (n-1) - 1| + |n-2|], [| 1 - 1| + 1 + |2 - (n-1)| + |n-n|] \} \\
&= \min \{ [| n-2| + |n-2|], [0 + 1 + |n-3| + 0] \} \\
d(ij, kl) &\leq \min \{ [2n-4], [n-2] \} \\
\text{Hence, } d(ij, kl) &\leq (n-2)
\end{aligned}$$

5 Double vertex graph of a Wheel

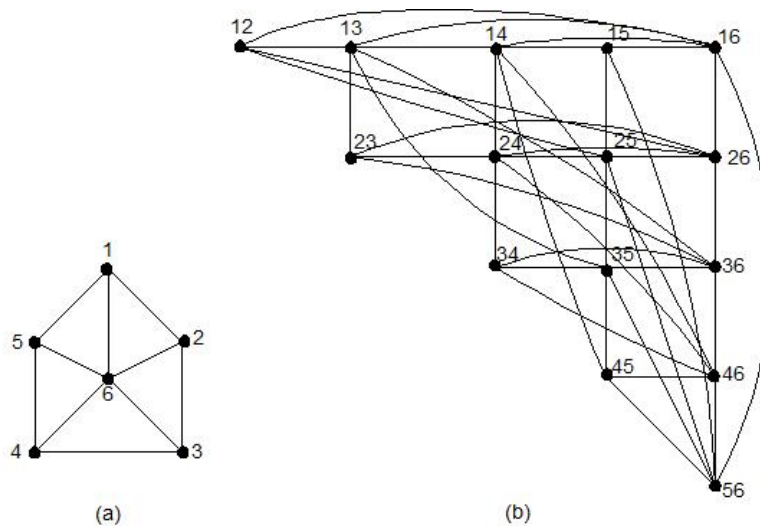


Figure 4: (a) Wheel W_6 (b) Double vertex graph of wheel $U_2(W_6)$

A *wheel graph* is a graph formed by connecting a single vertex to all vertices of a cycle. Wheel graph with n vertices is denoted as W_n . The double vertex graph of a wheel W_n is denoted as $U_2(W_n)$. In Figure 4, wheel W_6 and its double vertex graph $U_2(W_6)$ is shown. The number of vertices in $U_2(W_n)$ is $(n(n-1))/2$, the size of $U_2(W_n)$ is $m(n-2)$, where m is the size of W_n .

Theorem 5.1 The maximum distance between any two vertices in $U_2(W_n)$ is $d(u, v) \leq 4$.

Proof : Let $u, v \in U_2(W_n)$, $n \geq 7$. Based on the different positions of u and v , the distance between them is $d(u, v)$ takes different val-

ues as mentioned in the following cases :

Case (i): $d(u, v) = 1$

(a) $d(ij, in) = 1, \forall 1 \leq i \leq (n-1), 2 \leq j \leq n$

(b) $d(ij, jn) = 1, \forall 1 \leq i \leq (n-1), 2 \leq j \leq (n-1)$

(c) $d(1j, j(n-1)) = 1, \forall 2 \leq j \leq (n-2)$

(d) $d(1n, (n-1)n) = 1$

(e) $d(ij, kl) = 1 \begin{cases} \text{if } i = k \text{ and } |j - l| = 1 & \text{or} \\ \text{if } j = l \text{ and } |i - k| = 1 \end{cases}$

Case (ii): $d(u, v) = 2$

(a) $d(ij, il) = 2$ if $l \neq n$ and $|l - j| \geq 2, \forall 1 \leq i \leq (n-4)$

(b) $d(ij, kj) = 2$ if $|l - j| \geq 2$ and $4 \leq j \leq n$

(c) $d(ij, jl) = 2 \forall 2 \leq j \leq (n-1), |j - i| \geq 2$

Case (iii): $d(u, v) = 3$

This is obtained as follows

$$d(ij, kn) = d(ij, in) + d(in, ik) + d(ik, kn) \begin{cases} d(ij, in) = 1 & \text{by case(i) - (a)} \\ d(in, ik) = 1 & \text{by case(i) - (a)} \\ d(ik, kn) = 1 & \text{by case(i) - (b)} \end{cases}$$

$j \neq k, 1 \leq i \leq (n-1), 2 \leq j \leq (n-1), 2 \leq k \leq (n-1)$

$d(ij, kn) = 3$

Case (iv): $d(u, v) = 4$

For any $u = ij$ and $v = kl$

$$d(ij, kl) = d(ij, in) + d(in, ik) + d(ik, kn) + d(kn, kl) \begin{cases} d(ij, in) = 1 & \text{by case(i) - (a)} \\ d(in, ik) = 1 & \text{by case(i) - (a)} \\ d(ik, kn) = 1 & \text{by case(i) - (b)} \\ d(kn, kl) = 1 & \text{by case(i) - (a)} \end{cases}$$

$d(ij, kl) = 4$

Therefore by the above cases $d(u, v) \leq 4$

Hence the proof.

Observations 5.1 The maximum distance between any two vertices u and v in $U_2(W_n)$ are observed

(1) $d(u, v) \leq 2$ when $n = 4$ and 5

(2) $d(u, v) \leq 3$ when $n = 6$

6 CONCLUSION

The distance between any two vertices of double vertex graph of path, cycle and wheel was studied in this paper. This concept can

be extended to other graphical structures.

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