

COMMON FIXED POINT THEOREMS FOR D -OPERATOR PAIR IN CONE METRIC SPACE

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Abstract

In this paper ,we obtain some fixed point theorems for D -operator pair satisfying general contractive condition in cone metric space.This is an extension of occasionally weakly compatible mappings.

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1 Introduction

The study of common fixed point of mapping satisfying some contractive type condition for a pair of maps have been obtained by various authors. Jungck [9] gave the concept of compatible maps and showed that weakly commuting mappings are compatible, but converse is not true. During last few years several fixed point theorems have been obtained by various authors using this idea. Jungck further weakened the notion of compatibility by introducing weak compatibility. Al-Thagafi and Shahzad [4] defined the concept

of occasionally weakly compatible mapping which is more general than the concept weakly compatible maps. Recently, Deepmala and Pathak [1] have introduced D -operator pair which more general than occasionally weakly compatible mapping in metric space. Now, in this paper we prove some common fixed point theorem with some other different contractions in cone metric space.

2 Preliminaries

Definition 1. [2] Let E be a real Banach space. A subset P of E is called a cone if and only if

- (a) P is closed, nonempty and $P \neq \{0\}$
- (b) $a, b \in R, a, b \geq 0, x, y \in P$ implies $ax + by \in P$
- (c) $P \cap (-P) = \{0\}$

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ implies $y - x \in P$. A cone P is called normal if there is a number $k > 0$ such that for all $x, y \in E$ $0 \leq x \leq y$ implies $\|x\| \leq k\|y\|$.

The least positive number satisfying the above inequality is called normal constant of P , while $x \ll y$ stands for $y - x \in \text{int } P$ (interior of P).

Definition 2. [2] Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow E$ satisfying

- (a) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$
- (b) $d(x, y) = d(y, x)$ for all $x, y \in X$
- (c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

Then d is called a cone metric in X and (X, d) is called a cone metric in X and (X, d) is called a cone metric space. The concept of a cone metric space is more general than that of a metric space.

Definition 3. [2] Let (X, d) be a cone metric space. We say that $\{x_n\}$ is

- (a) A cauchy sequence if for every c in E with $c \gg 0$, there is N such that for all $n, m > N$, $d(x_n, x_m) \ll c$
- (b) A convergent sequence if for every c in E with $c \gg 0$, there is N such that for all $n > N$, $d(x_n, x) \ll c$ for some fixed x in X .

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X . It is known that $\{x_n\}$ convergent to x in X if and only if $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. The limit of a convergent sequence is unique provided P is a normal cone with normal constant k .

Definition 4. Let X be a set and let f, g be a two self mapping of X . A point $x \in X$ is called a coincidence point of f and g iff $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g .

Definition 5. Two self-maps f and g of a set X are occasionally weakly compatible iff there is a point $x \in X$ which is a coincidence point of f and g at which f and g commute.

Definition 6. [1] Let X be a non empty set and d be a function, $d : X \times X \rightarrow E$ such that

$$d(x, y) = 0 \text{ if and only if } x = y \text{ for all } x, y \in X \quad (1)$$

for a space (X, d) satisfying (1) and $A \subset X$, the diameter of A is defined by $diam(A) = \sup\{\max\{d(x, y), d(y, x)\} \text{ for all } x, y \in A\}$

Definition 7. [1] Let $f, g : X \rightarrow X$ be mappings. The pair (f, g) is said to be D -operator pair if there is a point u in X such that $u \in C(f, g)$ and $d(fgu, gfu) \leq R diam(PC(f, g))$ for some $R > 0$.

Definition 8. Let M be a nonempty subset of a cone metric space (X, d) . The set of best M -approximants to $u \in X$, denoted $P_M(u)$ is defined by

$$P_M(u) = \{y \in M : d(y, u) = dist(u, M)\}$$

where $dist(u, M) = inf\{d(x, u) : x \in M\}$.

3 Main Results

In this section, we prove some fixed point theorems for D -operator pair in cone metric space.

Theorem 9. Let (X, d) be a cone metric space and P be a normal cone. Let $\phi : R^+ \rightarrow R^+$ be a control function such that ϕ is continuous, nondecreasing, satisfying the condition $\phi(2t) < 2\phi(t)$ and $\phi(t) = 0$ iff $t = 0$. Let $\psi : R^+ \rightarrow R^+$ be another function such that $\psi(t) < t$ for each $t > 0$. If (f, g) is a D -operator pair such that $f(X) \subseteq g(X)$ and satisfy the condition $\phi d(fx, fy) \leq \Psi M_\phi(x, y)$, where

$$M_\phi(x, y) = \left(\max \left\{ \phi(d(gx, gy)), \phi(d(gx, fx)), \phi d(gy, fy), \right. \right. \\ \left. \left. \frac{1}{2}\phi(d(gx, fy) + d(fx, gy)) \right\} \right) \quad (2)$$

for all $x, y \in X$. Then f and g have a unique common fixed point.

Proof. Since (f, g) is a D -operator pair, there is a point $u \in X$ and $R > 0$ such that $fu = gu$ and

$$d(fgu, gfu) \leq R \text{diam}(PC(f, g)) \quad (3)$$

First, we prove that $PC(f, g)$ is singleton. Suppose w and z be two distinct points in X such that $w = fu = gu$ and $z = fv = gv$ for some $u, v \in C(f, g)$. Then from (2) we obtain $\phi d(w, z) = \phi d(fu, fv) \leq \Psi M_\phi(u, v)$, where

$$M_\phi(u, v) = \left(\max \left\{ \phi(d(gu, gv)), \phi(d(gu, fu)), \phi d(gv, fv), \right. \right. \\ \left. \left. \frac{1}{2}\phi(d(gu, fv) + d(fu, gv)) \right\} \right)$$

Then we get,

$$\begin{aligned} 0 \leq \phi(d(w, z)) &\leq \Psi(\phi(d(fu, fv))) \\ &\leq \phi(d(fu, fv)) \\ &< \phi d(w, z) \end{aligned}$$

This is a contradiction. So $w = z$, i.e., $w = fu = gu = fv = gv = z$. Hence $PC(f, g)$ is singleton. i.e., $w = fu = gu$ is the unique

point of coincidence and $diam(PC(f, g)) = 0$ and from (3) we have $fgu = gfu$ for some point $u \in C(f, g)$. We claim fu is the unique common fixed point of f and g . We first assert that fu is the fixed point of f . If $ffu \neq fu$ then from (2) we have,

$$\begin{aligned}\phi(d(ffu, fu)) &\leq \Psi(M_\phi(fu, v)), \text{ where} \\ M_\phi(fu, v) &= \left(\max \left\{ \phi(d(gfu, gv)), \phi(d(gfu, ffu)), \phi(d(fv, gv)), \right. \right. \\ &\quad \left. \left. \frac{1}{2}\phi(d(gfu, fv) + d(ffu, gv)) \right\} \right) \\ &= \left(\max \left\{ \phi(d(ffu, fu)), \phi(d(ffu, ffu)), \phi(d(fv, fv)), \right. \right. \\ &\quad \left. \left. \frac{1}{2}(\phi(d(ffu, fu)d(ffu, fu)) \right\} \right) \\ &= \phi(d(ffu, fu))\end{aligned}$$

Therefore,

$$\begin{aligned}\phi(d(ffu, fu)) &\leq \Psi(\phi(d(ffu, fu))) \\ &\leq \phi(d(ffu, fu))\end{aligned}$$

Which is a contradiction. Hence $ffu = fu$ and $ffu = fgu = fu$. Thus fu is a common fixed point of f and g . For uniqueness, suppose that $u, v \in X$, such that $fu = gu = u$ and $fv = gv = v$ and $u \neq v$. Then (2) gives,

$$\begin{aligned}\phi d(u, v) &= \phi(d(fu, fv)) \\ &\leq \Psi(M_\phi(u, v)), \text{ where}\end{aligned}$$

$$\begin{aligned}M_\phi(u, v) &= \max \left\{ \phi(d(gu, gv)), \phi(d(gu, fu)), \phi(d(gv, fv)), \right. \\ &\quad \left. \frac{1}{2}\phi(d(gv, fv) + d(fu, gv)) \right\} \\ &= \phi(d(u, v))\end{aligned}$$

Therefore

$$\begin{aligned}\phi(d(u, v)) &\leq \Psi(\phi(d(u, v))) \\ &\leq \phi(d(u, v))\end{aligned}$$

Which is a contradiction. Therefore $u = v$ and hence the common fixed point of f and g is unique.

Example 10. Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0, \subset R\}$ and define $d : R \times R \rightarrow E$ by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha > 0$ is constant. Define $f, g : X \rightarrow X$ by, $f(x) = \frac{1+6x}{7}$ and $g(x) = \frac{1+8x}{9}$, for all $x \in X$. Then (X, d) is a cone metric space. Clearly, f and g are D -operator pair and satisfy condition (2).

Corollary 11. Let (X, d) be a cone metric space and P be a normal cone. Let $\phi : R^+ \rightarrow R^+$ be a function, satisfying the condition for each $\phi(t) < t$ for each $t > 0$. If (f, g) is a D -operator such that

$$d(fx, fy) \leq \phi(\max\{(d(gx, gy)), (d(gx, fy)), d(gy, fx), d(gy, fy)\}) \quad (4)$$

for all $x, y \in X$. Then f and g have a unique common fixed point.

Proof. In the theorem (9) replacing the condition (2) by (4) we get the result that f and g have a unique common fixed point.

Corollary 12. Let (X, d) be a cone metric space and P be a normal cone. If (f, g) is a D -operator pair such that

$$d(fx, fy) \leq ad(gx, gy) + b(\max\{(d(fx, gx), d(fy, gy)\}) \quad (5)$$

$$+ c(\max\{(d(gx, gy), d(gx, gx), d(gy, fy)\}) \quad (6)$$

for all $x, y \in X$, where $a, b, c > 0$, $a + b + c \leq 1$. Then f and g have a unique common fixed point.

Proof. In the theorem (9) replacing the condition (2) by (5) we get the result that f and g have a unique common fixed point.

4 Application to best approximation

Theorem 13. Let (X, d) be a cone metric space. Suppose that $u \in X$, f and g satisfy inequality (2) in theorem (9). f leaves g -invariant compact subset M of a closed subspace gX as invariant. For each $b \in P_M(u)$, let $d(x, fb) < d(x, gb)$ and $fb \in P_M(u)$. If f and g are D -operator pair, then u has a best approximation in M which is also a common fixed point of f and g .

Proof. Let $u \in F(f) \cap F(g)$. Since M is a compact subset of gX , $P_M(u) \neq \emptyset$. To prove that $f(P_M(u)) \subset g(P_M(u))$, assume the contrary. Then there exist $b \in P_M(u)$ with $fb \notin g(P_M(u))$. Now,

$$\begin{aligned} d(u, gb) &= \text{dist}(u, M) \\ &< d(u, fb) \\ &< d(u, gb) \end{aligned}$$

Which is a contradiction. Hence $f(P_M(u)) \subset g(P_M(u))$. Now $f(P_M(u))$ being a closed subset of a complete cone metric space, it is complete. Hence all conditions of theorem (9) being satisfied we get $P_M(u) \cap F(f) \cap F(g)$ is singleton.

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