

## Prime Distance Labeling of Some Families of Graphs

A. Parthiban<sup>1,\*</sup> and N. Gnanamalar David<sup>1</sup>

<sup>1</sup> Department of Mathematics,

Madras Christian College, Chennai - 600 059, India

{mathparthi, ngdmcc}@gmail.com

### Abstract

A graph  $G$  is a prime distance graph if there exists a one-to-one labeling of its vertices  $f : V(G) \rightarrow \mathbb{Z}$  such that for any two adjacent vertices  $u$  and  $v$ , the integer  $|f(u) - f(v)|$  is a prime. So  $G$  is a prime distance graph if and only if there exists a prime distance labeling of  $G$ . For a graph  $G$  with vertex set  $V(G) = V$  and edge set  $E(G) = E$ , the Mycielskian of  $G$  is the graph  $\mu(G)$  with vertex set  $V \cup V' \cup \{w\}$ , where  $V' = \{x' : x \in V\}$ , and the edge set  $E \cup \{xy' : xy \in E\} \cup \{y'u : y' \in V'\}$ . In this paper we establish prime distance labeling of some Mycielskian graphs and powers of Mycielskian graphs. A graph is said to be triangle-free if no two adjacent vertices are adjacent to a common vertex. In this paper we also consider prime distance labeling for triangle-free graphs and conjecture that every triangle-free graph with  $\chi(G) \leq 4$  admits a prime distance labeling.

**AMS Subject Classification:** 05C15

**Key Words and Phrases:** Prime Distance Graphs, Prime Distance Labeling, Mycielskian Graphs, Triangle-free Graphs, Coloring, Chromatic Number

---

\*Research scholar

## 1 Introduction

All graphs in this paper are simple, finite and undirected. The distance graph, first introduced by Eggleton et al. in 1985 [1], is motivated by the well-known Hadwiger-Nelson plane coloring problem which asks for the minimum number of colors needed to color all points of the plane such that points at unit distances receive distinct colors. If  $D$  is a subset of the set of positive integers, then the integer distance graph  $G(Z, D)$  is defined to be the graph with vertex set  $Z$ , where two vertices  $u$  and  $v$  are adjacent if and only if  $|u - v| \in D$ . The prime distance graph  $G(Z, P)$  is the distance graph with  $D = P$ , the set of all primes. It has been proved that the chromatic number  $\chi(G(Z, P)) = 4$ . Research in prime distance graphs has since focused on the chromatic number of  $G(Z, D)$  where  $D$  is a non-empty proper subset of  $P$ . Clearly these graphs are all infinite (non-induced) subgraphs of  $G(Z, P)$ . In this paper we consider finite subgraphs of  $G(Z, P)$ . In [4] Laison et al. defined that a graph  $G$  is a prime distance graph if there exists a one-to-one labeling of its vertices  $f : V(G) \rightarrow Z$  such that for any two adjacent vertices  $u$  and  $v$ , the integer  $|f(u) - f(v)|$  is a prime and called  $f$  a prime distance labeling of  $G$ , so a graph  $G$  is a prime distance graph if and only if there exists a prime distance labeling of  $G$ .

**Definition 1.** [1] (i) A graph is said to be triangle-free if no two adjacent vertices are adjacent to a common vertex.

(ii) A  $k$ -coloring  $c$  of a graph  $G$ , is a function  $c : V(G) \rightarrow \{1, 2, \dots, k\}$ . Such a coloring is said to be proper if no two adjacent vertices have the same color, and a graph is  $k$ -colorable if it has a proper  $k$ -coloring.

(iii) The chromatic number of a graph  $\chi(G)$  is the smallest  $k \in N$  such that  $G$  has a proper  $k$ -coloring.

(iv) A set of vertices  $\{v_1, \dots, v_n\}$  is independent if  $v_i$  is not adjacent to  $v_j$  for all  $i, j \in [n]$ .

We denote a path or a cycle on  $n$  vertices by  $P_n$  or  $C_n$  respectively.

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski [5] developed an interesting graph transformation as follows. For a graph  $G$  with vertex set  $V(G) = V$  and edge

set  $E(G) = E$ , the Mycielskian of  $G$  is the graph  $\mu(G)$  with vertex set  $V \cup V' \cup \{w\}$ , where  $V' = \{x' : x \in V\}$ , and the edge set  $E \cup \{xy' : xy \in E\} \cup \{y'u : y' \in V'\}$ . The vertex  $x$  is called the twin of the vertex  $x'$  (and  $x'$  is also called the twin of  $x$ ); and the vertex  $u$  is called the root of  $\mu(G)$ . Thus, if  $G$  has  $n$  vertices and  $m$  edges,  $\mu(G)$  has  $2n + 1$  vertices and  $3m + n$  edges. The purpose of this paper is to investigate the prime distance labeling of certain Mycielskian graphs.

## 2 Mycielskian Graphs

The sequence of graphs we will consider is obtained by starting with a single edge  $K_2$ , and repeatedly applying the following graph transformation [3]. Suppose we have a graph  $G$ , with  $V(G) = \{v_1, v_2, \dots, v_n\}$ . The Mycielski transformation of  $G$ , denoted  $\mu(G)$ , has for its vertex set the set  $\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, z\}$  for a total of  $2n + 1$  vertices. As for adjacency, we put  $x_i \sim x_j$  in  $\mu(G)$  if and only if  $v_i \sim v_j$  in  $G$ ,  $x_i \sim y_j$  in  $\mu(G)$  if and only if  $v_i \sim v_j$  in  $G$  and  $y_i \sim z$  in  $\mu(G)$  for all  $i \in \{1, 2, \dots, n\}$ .

When Mycielski transformation is applied to a graph  $G$  with at least one edge, it has been shown [3] that a graph  $\mu(G)$  is obtained with the property  $P : \chi(\mu(G)) = \chi(G) + 1$ .

**Theorem 2.** [7] *The complete graph  $K_n$  for  $n \geq 5$  admits no prime distance labeling.*

**Theorem 3.** [7] *Any graph  $G$  with  $V(G) \subseteq \mathbb{Z}$  and  $\chi(G) \geq 5$  does not admit a prime distance labeling.*

**Theorem 4.** *The Mycielskian graph  $\mu(G)$  of any graph  $G$  with  $\chi(G) \geq 4$ , does not admit a prime distance labeling.*

The proof follows from Theorem 3 and property  $P$ .

We call a graph  $G$  as a non-prime distance graph, if  $G$  does not admit any prime distance labeling.

**Theorem 5.** *The Mycielskian graph  $\mu(G)$  of any non-prime distance graph  $G$  is again a non-prime distance graph.*

The proof is a consequence of the fact that in a graph  $G$ , if a subgraph of  $G$  does not admit a prime distance labeling, so does the graph  $G$  as well.

**Remark 1.** While examining the problem of establishing prime distance labeling of Mycielskian of cycles, no particular pattern of labeling could be arrived at. Hence we are proposing the following conjecture.

**Conjecture 1.** The Mycielskian graph  $\mu(C_n)$  of any cycle  $C_n$  admits a prime distance labeling.

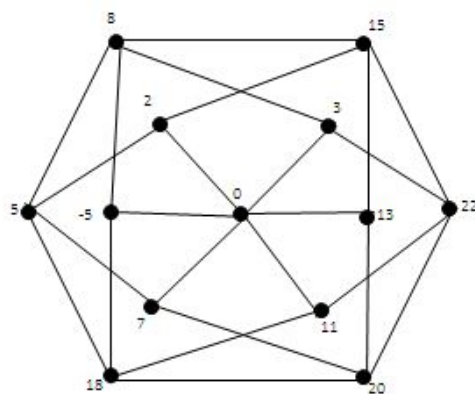


Figure 1: Prime distance labeling of  $\mu(C_6)$

**Remark 2.** A prime distance labeling of  $\mu(C_6)$  is shown in Fig. 1.

**Corollary 6.** The Mycielskian graph  $\mu(P_n)$  of any path  $P_n$  admits a prime distance labeling, if Conjecture 1 holds.

### 3 Powers of Mycielskian Graphs

**Definition 7.** The  $k$ -th power  $G^k$  of a graph  $G$  has the same vertex set as  $G$  and two distinct vertices  $u$  and  $v$  of  $G$  are adjacent

in  $G^k$  if and only if their distance in  $G$  is at most  $k$ . The graphs  $G^2$  and  $G^3$  are also referred to as square and cube of  $G$ , respectively.

**Lemma 1.** *The graph  $\mu(P_n)^k$ ,  $n, k \geq 2$ , contains a complete graph  $K_5$ .*

*Proof.* Consider the base case when  $n = k = 2$ . That is, we consider  $\mu(P_2)^2$ , the square of Mycielskian of path  $P_2$ . By the Definition 7, vertices at distance at most 2 will have an edge. This process results in a complete graph  $K_5$  as all the vertices in  $\mu(P_2)$ , at distance one or two, become adjacent in  $\mu(P_2)^2$ . The same argument holds good for any  $n, k > 2$ . Hence the lemma.  $\square$

**Theorem 8.** *Any  $k$ -th power ( $k \geq 2$ ) of Mycielskian graph  $\mu(P_n)$ ,  $n \geq 2$ , does not admit a prime distance labeling.*

The proof directly follows from Lemma 1 and Theorem 3. We can extend the same argument for cycles. Hence we have the following theorem.

**Theorem 9.** *Any  $k$ -th power ( $k \geq 2$ ) of Mycielskian graph  $\mu(C_n)$ ,  $n \geq 3$ , does not admit a prime distance labeling.*

## 4 Some Special Graphs

**Definition 10.** [2] Let  $T_{2n}$  denote the graph on the vertices  $u_i$ ,  $i = 1, 2, \dots, 2n$  with  $u_i$  non-adjacent to  $u_{i+n}$  for each  $i = 1, 2, \dots, n$  and all other pairs of vertices are adjacent. This unique  $(2n - 2)$ -regular graph on  $2n$  vertices is called the cocktail party graph.

**Definition 11.** The  $(n, k)$ -Turan graph, variously denoted  $T(n, k)$ ,  $T_{n,k}$  is the extremal graph on  $n$  vertices that contains no  $(k + 1)$ -clique for  $1 \leq k \leq n$ . In other words, the Turan graph has the maximum possible number of graph edges of any  $n$ -vertex graph not containing a complete graph  $K_{k+1}$ . The chromatic number of  $T(n, k)$  is  $k$ .

**Lemma 2.** *The cocktail party graph  $T_{2n}$  contains a complete graph  $K_n$ .*

The proof is immediate from the definition.

**Theorem 12.** *The cocktail party graph  $T_{2n}$ ,  $n \geq 5$  does not admit a prime distance labeling.*

The result follows from Lemma 2 and Theorem 3. Corresponding results for  $k$ -th power of  $T_{2n}$  and Turan graph  $T(n, k)$  can be stated as follows:

**Theorem 13.** (i) *Any  $k$ -th power ( $k \geq 2$ ) of cocktail party graph  $T_{2n}$ ,  $n \geq 3$  does not admit a prime distance labeling.*

(ii) *Turan graph  $T(n, k)$ , for  $n, k \geq 5$ , does not admit a prime distance labeling.*

## 5 Triangle-Free Graphs

In response to a question raised by Professor Lazio Babai who asked whether one could show that for every  $k \in N$ , there exists a triangle-free graph  $G$  with chromatic number  $\chi(G) = k$ , a family of graphs  $F$ , was first constructed [9]. Indeed,  $F$  does exhibit the above mentioned property. However, after further studying this family of graphs, one can easily realize that every triangle-free graph is an induced subgraph of some member of  $F$  [9].

**Construction** [9]:

We now recall the construction of the members of  $F$  iteratively. First, let  $T_1$  be the simple graph with one vertex. Now, given any graph,  $T_k$ , of this family, construct  $T_{k+1}$  by the following steps:

(1) Choose any independent set  $\{v_1, \dots, v_{k-1}\}$  of  $k-1$  vertices. (2) For this set of vertices, create a new vertex,  $u$ , and create an edge,  $e_i$ , between  $v_i$  and  $u$  for all  $i \in [k-1]$ . Do this for every independent set of  $k-1$  vertices. (3) Create a new vertex  $c$  and create an edge between  $c$  and each vertex  $u$  created in (2).

For this construction to make sense [9], we need some independent set of vertices of size  $k-1$  in the graph  $T_k$ . Note that the family of graphs,  $F = \{T_k : k \in N\}$  has only triangle-free members, and that the chromatic numbers of the graphs in  $F$  is unbounded.

**Definition 14.** [9] A graph  $H$  is an induced subgraph of another graph  $G$  if  $H$  can be obtained from  $G$  by removing a set

of vertices,  $S$ , and all edges associated to  $S$  in  $G$  from  $G$ . For any vertex  $v \in V(H)$ ,  $Hv$  is the induced subgraph formed by removing  $v$  and all edges associated to  $v$  from  $H$ .

It is easy to see that for all  $i \leq j$ ,  $T_i$  is an induced subgraph of  $T_j$ .

**Lemma 3.** [9] (i) For all  $k \in N$ ,  $T_k$  is triangle-free. (ii) For all  $k \in N$ ,  $\chi(T_k) = k$ .

**Theorem 15.** [9] Every member of  $F$  is triangle-free and for any  $k \in N$ ,  $\chi(T_k) = k$ .

**Theorem 16.** The graph  $T_k, k \geq 5$ , does not admit a prime distance labeling.

**Theorem 17** (Turan's Theorem). Let  $G$  be any graph with  $n$  vertices, such that  $G$  is  $K_{r+1}$ -free. Then the number of edges in  $G$  is at most  $\frac{r-1}{r} \cdot \frac{n^2}{2}$ . Equivalently, among the  $n$ -vertex simple graphs with no  $(r+1)$ -cliques,  $T(n, r)$  has the maximum number of edges.

By Turan's theorem, the  $n$ -vertex triangle-free graph with the maximum number of edges is a complete bipartite graph. This is a prime distance graph [4].

**Theorem 18.** [4] Every bipartite graph is a prime distance graph.

We propose the following conjecture which could be proved once one could characterize the triangle-free graphs with their chromatic numbers.

**Conjecture 2.** Every triangle-free graph  $G$  with  $\chi(G) \leq 4$  admits a prime distance labeling.

The result is trivial when  $\chi(G) = 1$  and  $\chi(G) = 2$  as a non-empty graph  $G$  is bicolourable if and only if  $G$  is bipartite. One can exhibit prime distance labeling of well known 3-chromatic and 4-chromatic triangle-free graphs, namely Bidiakis cube and Grotzsch graph, respectively in support of this conjecture.

**Acknowledgement:** The authors are very much grateful for the

valuable and detailed comments of the reviewer. And first author A. Parthiban acknowledges with gratitude the award (No.: F1-17.1/2014-15/RGNF-2014-15-SC-TAM-65968/ (SA-III/Website)) of Rajiv Gandhi National Fellowship for SC students by University Grants Commission, Government of India.

## References

- [1] J.A. Bondy, U.S.R. Murthy, *Graph Theory with Applications*, Elsevier, North Holland, New York, (1986).
- [2] D.A. Gregory, S. McGuinness, W. Wallis, Clique Partitions of the Cocktail Party Graph, *Discrete Mathematics*, **59** (1986) 267-273.
- [3] T. Jacobs, Fractional Colorings and the Mycielski graphs, *Master Thesis*, Portland State University (2006).
- [4] J. D. Laison, C. Starr, and A. Walker, Finite prime distance graphs and 2-odd graphs, *Discrete Mathematics*, **313** (2013) 2281-2291.
- [5] J. Mycielski, Sur le coloriage des graphs, *Colloq. Math.*, **3** (1955) 161-162.
- [6] A. Parthiban and N. Gnanamalar David, On Finite Prime Distance Graphs, Accepted for Publication in *Indian Journal of Pure and Applied Mathematics*, Springer (2017).
- [7] A. Parthiban and N. Gnanamalar David, On Prime Distance Labeling of Graphs, *Theoretical Computer Science and Discrete Mathematics*, Springer (2017), 238-241.
- [8] M. Rosenblum, Generalized Hermite polynomials and the Bose-like oscillator calculus, In: *Operator Theory: Advances and Applications*, Birkhäuser, Basel (1994), 369-396.
- [9] T. Zhang, Triangle Free Graphs and Their Chromatic Numbers, Technical Papers, Mathematics, University of Chicago (2008).



