

Oscillation of generalized quasilinear α -difference equations

V.Srimanju^{1*} and SK.Khadar Babu²

^{1,2} Vellore Institute of Technology,

Vellore - 632 014, Tamil Nadu, S.India.

¹*srimanjushc@gmail.com* ²*khadar.babu36@gmail.com*

September 22, 2017

Abstract

In this paper the authors establish conditions for the oscillatory behavior of solutions of generalized second order quasi linear α -difference equation.

AMS Subject Classification: 39A10, 39A11, 39A12

Key Words and Phrases: Oscillation, quasilinear

1 Introduction

The theory of difference equations is based on the operator Δ defined as $\Delta u(k) = u(k+1) - u(k)$, $k \in \mathbb{N} = \{0, 1, 2, \dots\}$. Even though many authors [1], [8] have suggested Δ as

$$\Delta u(k) = u(k+\ell) - u(k), \quad \ell \in (0, \infty). \quad (D)$$

In [4], authors labeled the operator Δ defined by (D) as Δ_ℓ and by defining its inverse Δ_ℓ^{-1} , many interesting results in number theory were obtained [2, 4, 5].

Jerzy Popenda [7], while discussing the behavior of solutions of a particular type of difference equation, defined Δ_α as $\Delta_\alpha u(k) =$

*Author of Correspondence

$u(k+1) - \alpha u(k)$. In [6] the authors have generalized the definition of Δ_α by $\Delta_{\alpha(\ell)}$ defined as $\Delta_{\alpha(\ell)}u(k) = u(k+\ell) - \alpha u(k)$ for the real valued function $u(k)$ and $\ell \in (0, \infty)$ and also obtained the solutions of certain types of generalized α -difference equations.

Recently, there has been an increasing interest in the study of the oscillatory behaviors to the solutions of nonlinear and linear difference equations [9, 10].

This paper is concerned with the oscillatory behavior of generalized quasilinear α -difference equations of the forms

$$\Delta_{\alpha(\ell)} \left(a((k-1)\ell + j) |\Delta_{\alpha(\ell)} v((k-1)\ell + j)|^{\gamma-1} \Delta_{\alpha(\ell)} v((k-1)\ell + j) \right) + q(k\ell + j) f(v(k\ell + j)) = 0, \quad k \in (0, \infty), \ell \in (0, \infty) \quad (1)$$

where $\Delta_{\alpha(\ell)}$ is the generalized α -difference operator defined by $\Delta_{\alpha(\ell)}v(k) = v(k+\ell) - \alpha v(k)$, $j = k - [\frac{k}{\ell}]\ell$, $a(k)$, $q(k)$ are real valued function, $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous, nondecreasing and $f(x) > 0$ for $x \neq 0$, $\gamma > 0$ and λ is a positive integer.

By a solution of equation (1) we mean a nontrivial function $v(k\ell + j)$ satisfying equation (1) for all $k\ell + j \geq 1$. A solution $v(k\ell + j)$ is said to be nonoscillatory if it is either eventually positive or eventually negative and oscillatory otherwise.

2 preliminaries

Definition 1. [3] For a real valued function $u(k)$, the generalized second order α -difference equation is defined by

$$v(k+2\ell) - 2\alpha v(k+\ell) + \alpha^2 v(k) = u(k), \quad k \in [0, \infty), \ell \in (0, \infty). \quad (2)$$

Definition 2. [6] (Product Formula). Let $u(k)$ and $v(k)$ be any two real valued functions. Then

$$\Delta_{\alpha(\ell)} \{u(k)v(k)\} = u(k+\ell)\Delta_{\alpha(\ell)}v(k) + \alpha v(k)\Delta_\ell u(k)$$

The following are assumed through out this paper

- (i) $q(k\ell + j)$ is a nonnegative real valued function with infinitely many positive terms,
- (ii) $-\infty < \sum_{k=1}^{\infty} q(k\ell + j) < \infty$,

- (iii) $R(k\ell + j, k_0\ell + j) = \sum_{s=k_0}^{k-1} \frac{1}{a^{1/\gamma}(s\ell + j)}, R(k\ell + j, k_0\ell + j) \rightarrow \infty$ and $R(k\ell + j) = R(k\ell + j, 1)$,
- (iv) $\left| \frac{f(x)}{x^\gamma} \right| \geq M > 0$ for $|x| > \delta > 0$

Lemma 3. Assume that conditions (i) and (iii) hold. If $v(k\ell + j)$ is a solution of equation (1) such that $v(k\ell + j) > 0$ for $k\ell + j \geq K\ell + j \geq 1$ then $\Delta_{\alpha(\ell)}v(k\ell + j) > 0$ for $k\ell + j \geq K\ell + j$.

Proof. Since $\{a(k\ell + j)|\Delta_\ell v(k\ell + j)|^{\gamma-1}\Delta_\ell v(k\ell + j)\}$ is non increasing by equation (1), we see that $\{\Delta_\ell v(k\ell + j)\}$ is eventually of constant sign, that is $\Delta_\ell v(k\ell + j) > 0$ for $k\ell + j \geq K\ell + j \geq 1$ or there is $K_1\ell + j > K\ell + j$ such that $\Delta_\ell v(k\ell + j) < 0$ for $k\ell + j > K_1\ell + j$. If $\Delta_\ell v(k\ell + j) < 0$ for $k\ell + j > K_1\ell + j$ we have

$$(a(k\ell + j))^{\frac{1}{\gamma}}\Delta_\ell v(k\ell + j) \leq (a(K_2\ell + j))^{\frac{1}{\gamma}}\Delta_\ell v(K_2\ell + j)$$

for $k\ell + j \geq K_2\ell + j \geq K_1\ell + j$.

Dividing the last inequality by $(a(k\ell + j))^{\frac{1}{\gamma}}$ and summing from $K_2\ell + j$ to $k\ell + j$, we obtain $v((k+1)\ell + j) \leq v(K_2\ell + j) + a(K_2\ell + j)^{\frac{1}{\gamma}}\Delta_\ell v(K_2\ell + j) \sum_{s\ell + j = K_2}^k \frac{1}{a(s\ell + j)^{1/\gamma}}$. As $(k\ell + j) \rightarrow \infty$ we see that $v(k\ell + j) \rightarrow \infty$, a contradiction. This completes the proof. $\square$

Lemma 4. If $u, v \geq 0$ then $u^y - v^y \leq y u^{y-1}(u - v)$ for $y \geq 1$ and $u^y - v^y \leq y v^{y-1}(u - v)$ for $0 < y < 1$.

3 Main Results

Theorem 5. Assume conditions (i) and (iii) hold. If

$$\liminf_{|x| \rightarrow \infty} \left| \frac{f(x)}{x^\gamma} \right| \geq d > 0, \quad (3)$$

there exists a positive real valued function $h(k\ell + j)$ such that

$$\lim_{k \rightarrow \infty} \sum_{s=1}^k h(s\ell + j)q(s\ell + j) = +\infty, \quad (4)$$

$$\text{and } \lim_{k \rightarrow \infty} \sum_{s=1}^k \frac{\alpha a(s\ell + j)(\Delta_{\alpha(\ell)} h((s+1)\ell + j))^{\gamma+1}}{h^{\gamma}((s+1)\ell + j)} < +\infty, \quad (5)$$

then every solution of equation (1) is oscillatory.

Proof. Let $v(k\ell + j)$ be a nonoscillatory solution of equation (1), say $v(k\ell + j) > 0$ for $k \geq K \geq 1$, since the proof for the case $v(k\ell + j) < 0$ for $k \geq K$ is similar. Let

$$z((k-1)\ell + j) = \frac{a((k-1)\ell + j)(\Delta_{\alpha(\ell)} v((k-1)\ell + j))^{\gamma}}{v^{\gamma}((k-1)\ell + j)}, \quad k \geq K+l$$

since $\Delta_{\alpha(\ell)} v((k-1)\ell + j) > 0$ by Lemma 3. Then

$$\begin{aligned} \Delta_{\alpha(\ell)}(z(k-1)\ell + j) &= -q(k\ell + j) \frac{f(v((k-l)\ell + j))}{v^{\gamma}((k-l)\ell + j)} \\ &\quad - \frac{\alpha a(k\ell + j)(\Delta_{\ell} v(k\ell + j))^{\gamma} \Delta_{\ell} v^{\gamma}((k-l)\ell + j)}{v^{\gamma}((k+1-l)\ell + j)v^{\gamma}((k-l)\ell + j)} \end{aligned} \quad (6)$$

$$\begin{aligned} \text{and hence } \Delta_{\alpha(\ell)}(z(k-1)\ell + j) &+ \frac{\alpha z(k\ell + j) \Delta_{\ell} v^{\gamma}((k-l)\ell + j)}{v^{\gamma}((k-l)\ell + j)} \\ &+ q(k\ell + j) \frac{f(v((n-l)\ell + j))}{v^{\gamma}((n-l)\ell + j)} = 0, \text{ for } k \geq K+l. \end{aligned} \quad (7)$$

Since $\Delta_{\ell} v(k\ell + j) > 0$ for $k \geq K+1$, $\lim_{k \rightarrow \infty} v(k\ell + j)$ exists. If $\lim_{k \rightarrow \infty} v(k\ell + j) = b$, then $\lim_{k \rightarrow \infty} \frac{f(v((n-l)\ell + j))}{v^{\gamma}((n-l)\ell + j)} = \frac{f(b)}{b^{\gamma}} > 0$. If $\lim_{k \rightarrow \infty} v(k\ell + j) = +\infty$ then, in view of condition (3), $\lim_{k \rightarrow \infty} \inf \frac{f(v((n-l)\ell + j))}{v^{\gamma}((n-l)\ell + j)} = d_1 \geq d > 0$. Let $b^* = \min\left(\frac{d}{2}, \frac{f(b)}{b^{\gamma}}\right)$, from (7) we have

$$\begin{aligned} \Delta_{\alpha(\ell)}(z(k-1)\ell + j) &+ \alpha z(k\ell + j) \frac{\Delta_{\ell} v^{\gamma}((k-l)\ell + j)}{v^{\gamma}((k-l)\ell + j)} \\ &+ b^* q(k\ell + j) \leq 0, \quad k \geq K+l. \end{aligned} \quad (8)$$

From the mean value theorem, we have

$$\begin{aligned} &\Delta_{\ell} v^{\gamma}((k-l)\ell + j) \\ &\geq \begin{cases} \gamma v^{\gamma-1}((k-l)\ell + j) \Delta_{\ell} v((k-l)\ell + j) & , \text{ if } \gamma \geq 1 \\ \gamma v^{\gamma-1}((k+1-l)\ell + j) \Delta_{\ell} v((k-l)\ell + j) & , \text{ if } 0 < \gamma < 1 \end{cases} \end{aligned} \quad (9)$$

for $k \geq K + l$. Hence from (8) and (9), we obtain for $\gamma \geq 1$

$$\Delta_{\alpha(\ell)}(z(k-1)\ell+j) + \alpha z(k\ell+j) \gamma \frac{\Delta_{\ell} v((k-l)\ell+j)}{v((k-l)\ell+j)} + b^* q(k\ell+j) \leq 0. \quad (10)$$

and for $0 < \gamma < 1$

$$\begin{aligned} & \Delta_{\alpha(\ell)}(z(k-1)\ell+j) + b^* q(k\ell+j) + \frac{\alpha \gamma z(k\ell+j)}{v^{\gamma}((k-l)\ell+j)} \\ & \times \Delta_{\ell} v^{\gamma-1}((k+1-l)\ell+j) \Delta_{\ell} v((k-l)\ell+j) \leq 0. \end{aligned} \quad (11)$$

Since $v(k\ell+j)$ is nondecreasing, we have from (10) and (11)

$$\Delta_{\alpha(\ell)}(z(k-1)\ell+j) + \gamma \frac{\alpha z(k\ell+j) \Delta_{\ell} v((k-l)\ell+j)}{v((k+1-l)\ell+j)} + b^* q(k\ell+j) \leq 0 \quad (12)$$

for $\gamma > 0$ and $k \geq K + l + 1$. Since $\Delta_{\alpha(\ell)}((a(k-1)\ell+j) \Delta_{\alpha(\ell)}(v(k-1)\ell+j)^k) \leq 0$, we have $(a(k-1)\ell+j) (\Delta_{\alpha(\ell)}(v(k-1)\ell+j))^{\gamma} \geq (a(k-1)\ell+j) (\Delta_{\alpha(\ell)}(v(k-1)\ell+j))^{\gamma}$ and so (12) gives for $k \geq N_1 \geq K + l + 1$

$$\Delta_{\alpha(\ell)}(z(k-1)\ell+j) + \alpha \gamma \frac{z(k\ell+j) z^{1/\gamma}(k\ell+j)}{a^{1/\gamma}((k-l)\ell+j)} + b^* q(k\ell+j) \leq 0. \quad (13)$$

Multiply (13) by $h(k\ell+j)$ and summing, we obtain

$$\begin{aligned} & b^* \sum_{s=K_1}^k h(s\ell+j) q(s\ell+j) \\ & \leq - \sum_{s=K}^k h(s\ell+j) \Delta_{\ell}(z(s-1)\ell+j) - \gamma \sum_{s=K_1}^k \frac{h(s\ell+j) (z(s\ell+j))^{\frac{\gamma+1}{\gamma}}}{a^{1/\gamma}((s-1)\ell+j)} \\ & = -h((k+1)\ell+j) z(k\ell+j) + h(K_1\ell+j) z((K_1-1)\ell+j) \\ & - \sum_{s=K_1}^k \frac{\gamma h(s\ell+j)}{a^{1/\gamma}((s-1)\ell+j)} \left[(z(s\ell+j))^{\frac{\gamma+1}{\gamma}} - \frac{\Delta_{\ell} h(s\ell+j)}{\gamma h(s\ell+j)} \right. \\ & \times a^{1/\gamma}((s-1)\ell+j) z(s\ell+j) + \frac{1}{\gamma} \left(\frac{\Delta_{\ell} h(s\ell+j)}{(\gamma+1)h(s\ell+j)} \right)^{\gamma+1} a^{\frac{\gamma+1}{\gamma}}((s-1)\ell+j) \left. \right] \\ & + \sum_{s=K_1}^k \frac{(\Delta_{\ell} h(s\ell+j))^{\gamma+1}}{(\gamma+1)^{\gamma+1} h^{\gamma}(s\ell+j)} (a(s-1)\ell+j). \end{aligned}$$

Using Lemma 4 we obtain

$$b^* \sum_{s=K_1}^k h(s\ell + j)q(s\ell + j) \leq h(K_1\ell + j)z((K_1 - 1)\ell + j) \\ + \sum_{s=K_1}^k \frac{(a(s - l)\ell + j)(\Delta_\ell h(s\ell + j))^{\gamma+1}}{(\gamma + 1)^{\gamma+1}h^\gamma(s\ell + j)}.$$

This gives contradiction to conditions (4) and (5) and the proof is complete. $\square$

Next assume that conditions (i) and (ii) are satisfied. Let $h_0(k\ell + j) = \sum_{i=k+1}^{\infty} q(i\ell + j)$ and $\left| \frac{f(x)}{x^\gamma} \right| \geq M > 0$ for $|x| \geq \delta > 0$.

Define $h_r(k\ell + j) = M^{\frac{\gamma+1}{\gamma}} \gamma \sum_{i=k+1}^{\infty} \frac{(h(r-1(i)))^{1+1/\gamma}}{a(i-l)^{1/\gamma}} + Mh_0(k\ell + j)$, $r \in (0, \infty)$ and $E(k\ell + j) = \sum_{i=1}^k \frac{1}{(a((i-l)\ell + j))^{1/\gamma}}$.

Theorem 6. Assume conditions (i), (ii), (iii) and (iv) hold. Further assume that one of the following conditions hold.

- (a) $\sum_{k=1}^{\infty} q(k\ell + j) = +\infty$, that is $h_0(0)$ does not exist;
- (b) there exists a positive integer j such that $h_n(k\ell + j)$ is defined for $n \in [0, r - 1]$ but $h_r(O)$ does not exist.

Then every solution of equation (1) is oscillatory.

Proof. Assume that $v(k\ell + j)$ is a non oscillatory solution of equation (1) with $v(k\ell + j) > 0$ for $k > K + l$. By Lemma 3, $b.\Delta_{\alpha(\ell)}v(k\ell + j) > 0$ for $k \geq N + l$. Now define

$$W((k - 1)\ell + j) = \frac{(a(k - 1)\ell + j)(\Delta_\ell(v((k - 1)\ell + j)))^\gamma}{v^\gamma((k - l)\ell + j)}$$

for $k \geq K + l$. Then from the proof of Theorem 5, we obtain

$$\Delta_{\alpha(\ell)}W((k - 1)\ell + j) + \alpha\gamma \frac{W^{1+\frac{1}{\gamma}}(k\ell + j)}{(a^{1/\gamma}(k - l)\ell + j)} \\ + Mq(k\ell + j) \leq 0, \quad k \geq K_1 \geq K + l + 1. \quad (14)$$

Summing the last inequality from K_1 to k , we obtain

$$\begin{aligned} W(k\ell + j) + (1 - \alpha) \sum_{k=1}^{K_1} w(r\ell + j) + \alpha\gamma \sum_{s=K_1}^k \frac{W^{1+\frac{1}{\gamma}}(s\ell + j)}{(a^{1/\gamma}(s-l)\ell + j)} \\ + M \sum_{s=K_1}^k q(s\ell + j) \leq \alpha W((K_1 - 1)\ell + j). \end{aligned} \quad (15)$$

If $\sum_{s=K_1}^{\infty} \frac{W^{1+\frac{1}{\gamma}}(s\ell + j)}{a^{1/\gamma}((s-l)\ell + j)} = +\infty$ then $\lim_{k \rightarrow \infty} \left(1 + \frac{W(k\ell + j)}{\alpha\gamma \sum_{s=K_1}^k \frac{W^{1+\frac{1}{\gamma}}(s\ell + j)}{(a^{1/\gamma}(s-l)\ell + j)}} \right) \leq 0$

or $\lim_{k \rightarrow \infty} \sup \frac{W(k\ell + j)}{\gamma \sum_{s=K_1}^k \frac{W^{1+\frac{1}{\gamma}}(s\ell + j)}{a^{1/\gamma}((s-l)\ell + j)}} \leq -1$, which contradicts $W(k\ell + j) > 0$.

Hence $\sum_{s=K_1}^{\infty} \frac{W^{1+\frac{1}{\gamma}}(s\ell + j)}{a^{1/\gamma}((s-l)\ell + j)} < \infty$. On the other hand from (14) we have $\Delta_\ell(W(k-1)\ell + j) \leq 0$ which imply $\lim_{k \rightarrow \infty} W(k\ell + j) = c \geq 0$. If $c > 0$ then it would imply that $\sum_{s=K_1}^{\infty} \frac{1}{a^{1/\gamma}((k-l)\ell + j)} < \infty$, a contradiction. Hence $\lim_{k \rightarrow \infty} W(k\ell + j) = 0$.

(a) If $h_0(0) = +\infty$, then from (15) we have $W(k\ell + j) \leq \alpha W((K_1 - 1)\ell + j) - M \sum_{s=K_1}^k q(s\ell + j)$. This gives a contradiction.

(b) If $h_n(k\ell + j)$ is defined for $n \in [0, r-1]$, letting $k \rightarrow \infty$ in (15), we have

$$W(k\ell + j) \geq \alpha\gamma \sum_{s=k+1}^{\infty} \frac{W^{1+\frac{1}{\gamma}}(s\ell + j)}{a^{1/\gamma}((s-l)\ell + j)} + Mh_0(k\ell + j). \quad (16)$$

and hence $W(k\ell + j) \geq Mh_0(k\ell + j)$. If $r = 1$ then

$$\sum_{s=k+1}^{\infty} \frac{h_0^{1+\frac{1}{\gamma}}(s\ell + j)}{a^{1/\gamma}((s-l)\ell + j)} \leq \frac{1}{M^{1+\frac{1}{\gamma}}} \sum_{s=k+1}^{\infty} \frac{W^{1+\frac{1}{\gamma}}(s\ell + j)}{a^{1/\gamma}((s-l)\ell + j)} < \infty$$

which contradicts $h_1(0) = +\infty$. If $r > 1$, then

$$\begin{aligned} h_1(k\ell + j) &= M^{1+\frac{1}{\gamma}}\gamma \sum_{i=k+1}^{\infty} \frac{h_0(i\ell + j)}{(a^{1/\gamma}(i-l)\ell + j)} + Mh_0(k\ell + j) \\ &\leq \alpha\gamma \sum_{i=k+1}^{\infty} \frac{W^{1+\frac{1}{\gamma}}(i\ell + j)}{(a^{1/\gamma}(i-l)\ell + j)} + Mh_0(k\ell + j) \leq W(k\ell + j). \end{aligned}$$

In a similar fashion, we can prove $W(k\ell + j) \geq h_k(k\ell + j)$ for

$$n \in [1, r-1]. \text{ Therefore } \sum_{i=k+1}^{\infty} \frac{h_{r-1}^{1+\frac{1}{\gamma}}(i\ell + j)}{(a^{1/\gamma}(i-l)\ell + j)} \leq \sum_{i=k+1}^{\infty} \frac{W^{1+\frac{1}{\gamma}}(i\ell + j)}{(a^{1/\gamma}(i-l)\ell + j)} < \infty$$

which contradicts (b). $\square$

References

- [1] R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York (2000).
- [2] A. Benevatho Jaison and S. K. Khadar Babu, Oscillation for generalized first order nonlinear difference equations, *Global J. of Pure and Applied Mathematics*, **12(1)** (2016), 51-54.
- [3] G.Britto Antony Xavier, et. al., Finite Series and Complete Solution of Second Order α -Difference Equation, *Int. J. of Pure and Applied Mathematics*, **90(2)** (2014), 177-188.
- [4] M. Maria Susai Manuel, G. Britto Antony Xavier, E. Thandapani, Theory of generalized difference operator and its applications, *Far East J. of Math. Scis.*, **20(2)** (2006), 163-171.
- [5] M. Maria Susai Manuel, et. al., Qualitative properties of solutions of certain class of difference equations, *Far East Journal of Mathematical Sciences*, **23(3)** (2006), 295-304.
- [6] M. Maria Susai Manuel, et. al., Qualitative Properties of Solutions of Riccati's α -Difference Equations, *Int. J. of Differential Equations and Appl.*, **24(6)** (2011), 943-954.
- [7] J. Popena and B. Szmanda, On the oscillation of solutions of certain difference equations, *Demonstratio Mathematica*, **17(1)** (1984), 153-164.
- [8] Ronald E. Mickens, *Difference Equations*, Van Nostrand Reinhold Company, New York (1990).
- [9] P. J. Wong and R. P. Agarwal, Oscillation theorems for certain second order nonlinear difference equations, *J. Math. Anal. Appl.* **204** (1996), 813-829.
- [10] Y. Yang and W. Zhang, Oscillation for second order superlinear difference equations, *J. Math. Anal. Appl.* **189** (1995), 631-639.

