

DOMATIC NUMBER OF A BUTTERFLY NETWORK

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Abstract

The domatic number of G is the maximum number k such that $V(G)$ can be partitioned into k disjoint dominating set. The problem of finding the domatic number problem is NP-complete for any arbitrary graph. The domatic number problem arises in communication networks. Motivated by this fact, butterfly network is focussed as it plays a vital role in computer architecture and communication networks. In view of this a study on domatic number of butterfly network is investigated and studied. Moreover, the diameter and domination number of butterfly network is given in terms of domatic number.

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Key Words and Phrases: Dominating set, Minimal dominating set, Domatic partition, Domatic number, Diameter, Domination number.

1 Introduction

Graph theory is one of the most important and fastest growing area in various fields. The most common representation of a graph is by means of a diagram. A graph can be used to represent any physical situation involving discrete objects and a relationship among them. The major role of graph theory in computer applications is the development of graph algorithms. Also, graph theory is used in research areas of computer science such as data mining, image segmentation, clustering, image capturing, networking etc.

Domination is an important area of study. Berge and Ore were the first to define the dominating set [1, 2]. There are several applications in domination such as facility and guard location problem, surveillance system and coding theory. One of the most often discussed is the communication network, where a dominating set represent a set of cities as transmission stations can transmit messages to every city in the network. Hence, a transmitting group in the network is a dominating set [3]. The problem of finding a domatic number was first introduced by Cockayne and Hedetniemi [4]. The complexity of domatic number have been widely studied. Also finding the domatic partition of optimal size is computationally hard.

In recent years graph theory has established itself as an important mathematical tool in a wide variety of subjects and provides a fundamental tool for analyzing networks. In line of thought we would like to focus on interconnection networks. An interconnection network consists of set of processors and communication links for the data transmission. It is used for exchanging the data between two processors in a multistage network. A topology is defined in terms of a number of parameters. Earlier work on topological structures and analysis of interconnection networks has been investigated by Xu [5]. A number of interconnection networks have been suggested in literature. One of the most attractive and fascinating architecture is the butterfly network. The butterfly has a very good symmetry in structure. This network has been proposed as suitable topologies for parallel computers, due to their interesting architecture.

2 Basic Concept

Consider the graphs are finite, simple and undirected graph, without loops or multiple edges. Let $G = (V, E)$ be a graph, where $V(G)$ is the vertex set and $E(G)$ is the edge set. A maximal complete sub graph which is a clique, is a set of vertices every pair of which are adjacent. A subset D of $V(G)$ is called a dominating set if every vertex in $V - D$ is adjacent to at least one vertex in D . A subset S of V is said to be a minimal dominating set if $S - \{u\}$ is not a dominating set for any $u \in S$. The domination number $\gamma(G)$ of G is the smallest size of a dominating set of G . The diameter denoted by $diam(G)$ is the length of any longest geodesic. A vertex v of degree zero in G is called an isolated vertex of G .

3 Overview of Paper

Butterfly graphs are well known for FFT network which perform Fast Fourier transform efficiently. As the network is the current topic of the research, domatic number of butterfly networks is focussed. The main purpose of the paper is to study the domatic number of a butterfly graph and its characterization according to the number of vertices. Also the relationship between diameter and domatic number is observed in terms of domatic number.

4 Domatic Number

A domatic partition is the partition of vertices into disjoint dominating sets. The maximum number of disjoint dominating sets into which the vertex set of a graph G can be partitioned is called the domatic number of G and denoted by $dom(G)$. It is proved that for any graph $dom(G) = 1$, if and only if G has an isolated vertex. Also, a graph G is called domatically full if $dom(G) = \delta(G) + 1$. Figure 1 shows the domatic number, minimal dominating set and domination number.

The domatic partition of $V(G)$ is $D_1 = \{V_2, V_4, V_7\}$, $D_2 = \{V_3, V_5\}$, and $D_3 = \{V_1, V_6\}$, which is of size 3. Hence the domatic number = 3. D_1 and D_2 are minimum dominating set. The domination number of $\gamma(G) = 2$. Here D_1 , D_2 , and D_3 are all minimal

dominating set.

Lemma 1. [4] Every graph G satisfies $D(G) \geq 1$, and unless G contains an isolated node, $D(G) \geq 2$.

Lemma 2. [4] Let δ denote the minimum degree of a node in the graph, then $D(G) \leq \delta + 1$.

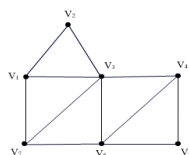


Figure 1: The Domatic number of G is 3, i.e. $dom(G)=3$.

5 Domatic number of butterfly network

The set of nodes V of an n -dimensional butterfly correspond to pairs $[w, i]$, where i is the dimension or level of a node ($0 \leq i \leq n$) and w is an n -bit binary number that denotes the row of the node. Two nodes $\langle w, i \rangle$ and $\langle w', i' \rangle$ are linked by an edge if and only if $i' = i + 1$ and either:

1. w and w' are identical, or
2. w and w' differ in precisely the i^{th} bit.

The edges in the network are undirected. An n -dimensional butterfly is denoted by BF_n . From its definition, the butterfly network has $(n + 1)2^n$ vertices, because $BF(n)$ has $n + 1$ levels and there are 2^n vertices in every level. Each vertex on levels 0 and n is of degree 2, otherwise, every vertex of degree 4.

The main results are as follows:

Theorem 1. Let G be a butterfly graph of dimension n . For every $n \in \mathbb{N}$, the domatic number of a graph holds the following conditions.

1. Each domatic partition is a minimal dominating set.
2. Each domatic partition has a maximal clique k_2 .

Proof. (i) **To Prove : Each domatic partition is a minimal dominating set.**

Let $D = \{D_1, D_2, \dots, D_k\}$ be the domatic partition of vertices into k disjoint dominating sets. If D_i is the dominating set of G , then D_i dominates each vertex $v_i \in D_i$ on the graph G . Clearly, every vertex $v_i \in D_i$ dominates atleast two vertices in G such that no two vertices are adjacent. In each dominating set D_i , removal of a vertex leaves the set that does not dominate G .

$\Rightarrow D_i$'s are minimal dominating set.

Thus in particular, as every D_i is a dominating set, each domatic partition possesses a minimal dominating set.

(ii) **To prove: Each domatic partition has a maximal clique.**

Construct a partition of vertices $v(G)$ into k disjoint dominating sets such that $D = \{D_1, D_2, \dots, D_k\}$.

$\Rightarrow$ each dominating set D_i forms a domatic partition.

Clearly, each vertex $v_i \in D_i$ meets all the remaining vertices in G which induces a complete subgraph.

$\Rightarrow$ each domatic partition has a maximal clique.

This completes the proof. $\square$

Theorem 2. Let G be a butterfly graph. The $\text{dom}[BF_n] = 2$ if $n = 1, 2$.

Proof. Let G be a butterfly network of dimension $n = 1, 2$. Let D_1 and D_2 be the two dominating sets of G with the least and greatest number of vertices. Here, the cardinality of D_1 and D_2 differ by at least one.

Case (i): For $n=1$.

Consider $u \in D_1$ is adjacent to at least 2 vertices in G . Then, $G - D_1 \in V(G)$ does not dominate G . i.e, the number of vertices of level n required to dominate all vertices of level $n - 1$.

$\Rightarrow$ there exists D_1 and D_2 , where D_1 dominates the vertices of level 0 and D_2 dominates the vertices of level 1. By Theorem 1, the graph G has a minimal dominating set and a maximal clique k_2 .

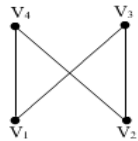
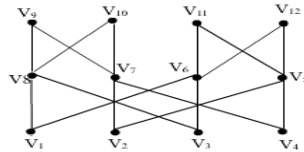
Case (ii): For $n=2$.

The number of vertices of level $n - 1$ required to dominate all the vertices of level $n - 2$ and n .

$\Rightarrow D_1$ and D_2 forms a domatic partition of size 2.

Figure 2 is the butterfly network of dimension 1. The Domatic par-

titions are $D_1 = \{V_1, V_2\}$ and $D_2 = \{V_3, V_4\}$. Hence $\text{dom}[BF_1] = 2$. Figure 3 is the butterfly network of dimension 2. The Domatic partitions are $D_1 = \{V_5, V_6, V_7, V_8\}$ and $D_2 = \{V_1, V_2, V_3, V_4, V_9, V_{10}, V_{11}, V_{12}\}$. Hence $\text{dom}[BF_2] = 2$.

Figure 2: $\text{dom}[BF_1] = 2$ Figure 3: $\text{dom}[BF_2] = 2$

This completes the proof. $\square$

Theorem 3. Let G be a butterfly graph of dimension $n \geq 3$. Then $\text{dom}[BF(n)] = 3$.

Proof. Let G be a butterfly graph of dimension $n \geq 3$ with a vertex set and an edge set. Consider $D = \{D_1, D_2, \dots, D_k\}$ be the k disjoint dominating set.

Assume that $\text{dom}[BF(n)] > 3$.

Let $D = \{D_1, D_2, D_3, D_4\}$ be the 4 disjoint dominating set such that no two vertices in each D_i 's are adjacent. Each D_i has at least 2 vertices and the other dominating set is greater than the defined set.

Each D_i has at least 2 vertices and the other dominating set is greater than the defined set. The vertex v_i in D_1 is adjacent to v_j in D_2 where $1 \leq i, j \leq n$, such that $|D_1 \cap D_2| = \phi$ and $|D_2 \cap D_3| = \phi$ but $|D_1 \cap D_4| = \phi$.

$\Rightarrow D_4$ does not exist which is not a dominating set, as the vertices from level 0 to level 1 is dominated by D_1, D_2 and D_3 such that each dominating set has a vertex with maximum 4 and minimum degree 2. Hence $\text{dom}[BF(n)] \not\geq 3$.

Assume that $\text{dom}[BF(n)] > 3$. By Theorem 2, $\text{dom}[BF(n)] = 2$ for $n = 1, 2$. Since $n \geq 3$, $\text{dom}[BF(n)] \not\geq 3$. Therefore, $\text{dom}[BF(n)] = 3$. $\square$

Theorem 4. Let G be a butterfly graph of dimension $n \geq 2$. Then $2 \leq \text{dom}[BF(n)] \leq \delta + 1$, if and only if each partition of dominating set has even number of vertices.

Proof. Let G be a butterfly graph of dimension $n \geq 2$. Assume that $2 \leq \text{dom}[BF(n)] \leq \delta + 1$.

To Prove: G has even number of vertices.

Consider $D = \{D_1, D_2, \dots, D_k\}$ be the k - dominating set of BF_n . The number of vertices of $BF(n)$ in each level is 2^n .

$\Rightarrow$ At least two vertices in each level is a dominating set.

$\Rightarrow$ Clearly, the cardinality of each dominating set D_i of G is even.

$\Rightarrow G$ has even number of vertices.

Conversely, Assume that each D_i has even number of vertices.

To Prove: $2 \leq \text{dom}[BF(n)] \leq \delta + 1$

Now consider the butterfly network of dimension greater than 4. Let D_i be the dominating set. Since the number of vertices of even degree is always even, each domatic partition has even number of vertices. As the minimum degree of each vertex is 2, then

$$\text{dom}[BF_n] \leq \delta + 1 \quad (1)$$

By Theorem 2

$$\text{dom}[BF_n] = 2, \text{ if } n = 1, 2. \quad (2)$$

From (1) and (2), $2 \leq \text{dom}[BF(n)] \leq \delta + 1$.

This completes the proof. $\square$

Theorem 5. Let G be a butterfly graph of dimension $n \geq 3$. Then $\text{dom}[BF_n] = 3$, if and only if $\gamma(BF_n) \geq \text{dom}[BF_n] + 1$.

Proof. Consider a butterfly graph G of dimension $n \geq 3$. Assume that $\gamma(BF_n) \geq \text{dom}[BF_n] + 1$.

Case (i) : Suppose $\text{dom}[BF_n] < 3$.

$\Rightarrow$ there exist a domatic partition of exactly 2 dominating set.

$\Rightarrow$ each dominating set with minimum cardinality such that $|D_1| = |D_2|$ and $|D_1| > |D_2|$. Since $n \geq 3$, number of vertices of G is $(n+1)2^n$.

$\Rightarrow \gamma(BF_n) \neq 3$.

Hence $\text{dom}[BF_n] \neq 2$. Therefore $\text{dom}[BF_n] > 3$.

Case(ii): Suppose $\text{dom}[BF_n] = 3$.

Since $n \geq 3$, number of vertices of G is $(n+1)2^n$, $\text{dom}[BF(n)] \leq \delta + 1$.

$\Rightarrow \text{dom}[BF_n] \not\geq 3$. Therefore $\text{dom}[BF_n] = 3$.

Conversely, assume that $\text{dom}[BF_n] = 3$.

To Prove: $\gamma(BF_n) \geq \text{dom}[BF_n] + 1$

Suppose $\gamma(BF_n) < \text{dom}[BF_n] + 1$.

This completes the proof. $\square$

Observation 1. Let G be a butterfly network of dimension $n \geq 3$. Then $\text{diam}(BF_n) = \text{dom}(BF_n) + (2n - 3)$.

In the literature survey, researchers have defined set inclusion for the graph. But here we, give the set inclusion for the considering problem to find the domatic number, diameter and domination number in the following observation.

Observation 2. Let G be a butterfly network of n dimension. For every $n \in N$, $\text{dom}(BF_n) \leq \text{diam}(BF_n) \leq \gamma(BF_n)$.

6 Conclusion

In this paper, we have identified the domatic number problem of butterfly network and the relationship between diameter and domination number is expressed in domatic number. The problem is worth considering for Cayley and Non-Cayley graphs. This problem is under investigation for certain architectures like silicate network and oxide network.

References

- [1] C. Berge, *Graphs and Hypergraphs*, North Holland, Amsterdam, 1973.
- [2] O. Ore, *Theory of Graphs*, Amer. Maths. Soc. Colloq. Pub., 38 (1962).
- [3] H. Kaplan, R. Shamir, *The domatic number problem on some perfect graph families*, Information processing letters 49 (1994), 51-56.
- [4] E.J. Cockayne, and S.T. Hedetnieme, *Towards a theory of domination in graphs*, Networks 7 (1977).
- [5] J. Xu, *Topological structures and analysis of interconnection network*, Kluwer academic publishers (2001).

