

THE IMPACT OF CERTAIN GRAPH OPERATIONS ON CONSTRUCTING GRACEFUL GRAPHS

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Abstract

A graceful labeling of a graph G with n edges is an injection $f : V(G) \rightarrow \{0, 1, 2, \dots, n\}$ with the property that when each edge uv is assigned the label $|f(u) - f(v)|$ then the resulting edge labels are distinct. The characterization of graceful graphs is one of the most difficult problems in graph theory. In this paper, we study the impact of two graph operations, switching of a vertex in a graph and t -levels shadow graph of a graph in obtaining graceful graphs. More precisely we prove the following results.

1. Switching a vertex of the non-graceful graph $\bar{K}_n + 2K_2$, for $n \geq 2$ is graceful.
2. For $t \geq 2$, the t -levels shadow graph of any α -labeled graph admits α -labeling.

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1 Introduction

All the graphs considered in this paper are finite and simple graphs. In 1963, at the Smolenice symposium, Ringel [5] posed his cele-

brated conjecture which states that K_{2n+1} , the complete graph on $2n + 1$ vertices, can be decomposed into $2n + 1$ isomorphic copies of a given tree with n edges. In 1965, Kotzig [4] also conjectured that the complete graph K_{2n+1} can be cyclically decomposed into $2n + 1$ copies of a given tree with n edges. Motivated by these two conjectures, Rosa [6] introduced ‘classical’ graph labelings in 1967 and used these labelings to investigate the cyclic decomposition of complete graphs. One of the labelings, called the β -labeling, introduced by Rosa [6] was later called graceful labeling by Golomb [3] and now this is the term most widely used. A graceful labeling of a graph G with n edges is an injection $f : V(G) \rightarrow \{0, 1, 2, \dots, n\}$ with the property that when each edge uv is assigned the label $|f(u) - f(v)|$ then the resulting edge labels are distinct. A graph which admits a graceful labeling is called a graceful graph. In [6], Rosa proved that if a graph G with n edges admits graceful labeling, then the complete graph K_{2n+1} can be decomposed into $2n + 1$ isomorphic copies of G . Thus, by Rosa’s theorem, if every tree is graceful then the Ringel’s conjecture is true. This observation led to the birth of Ringel-Kotzig-Rosa conjecture, popularly called the Graceful Tree Conjecture, which states that every tree is graceful. Graceful Tree Conjecture is one of the most difficult conjectures in graph theory, it remains open for five decades.

Both in the context of graph decomposition and its utility in the application problems, understanding the structural characterization of graceful graphs is very much important. The characterization of graceful graphs is one of the hardest problems in graph theory. In fact, a very few general results are proved towards characterization of graceful graphs. The following two results are the necessary conditions for a graph to be graceful.

Theorem 1 (Golomb [3]). *Let G be a graceful graph with n vertices and e edges. Let the vertices be partitioned into two sets E and O having respectively the vertices with even and odd labels. Then the number of edges connecting the vertices in E with the vertices in O is exactly $\frac{e+1}{2}$.*

Theorem 2 (Rosa [6]). *Let G be an Eulerian graph. If $|E(G)|$ is equivalent to 1 or 2 (mod 4), then G does not have a graceful labeling.*

Sethuraman and Elumalai [7] have shown that any graph can be embedded in a graceful graph. This implies that it is not possible to characterize graceful graphs by forbidding any graph. On the other hand, number of sufficient conditions are obtained using graph operations. For an exhaustive survey on graph labeling refer excellent survey by Gallian [2]. In this direction, we investigate the impact of two graph operations, switching of a vertex in a graph and t -levels shadow graph of a graph in obtaining graceful graphs.

Vertex switching of a vertex v in a graph G is the graph obtained by removing all the edges incident with v and joining all the non-adjacent vertices of v to v and it is denoted by $G_{\bar{v}}$. The t -levels shadow graph of a graph G , denoted $S_t(G)$ is obtained by taking $t \geq 2$ copies of G and joining each vertex v_{ij} in G_i to the copies of its adjacent vertices in G_{i+1} , for $1 \leq j \leq n$ and $1 \leq i \leq t$, where $n = |V(G)|$.

We surprisingly note that, switching a vertex in certain non-graceful graphs results in graceful graphs. More precisely, we prove the following result. Switching a vertex of the non-graceful graph $\bar{K}_n + 2K_2$ for $n \geq 2$ is graceful. Further we prove that for $t \geq 2$, the t -levels shadow graph of every α -labeled graph admits α -labeling, a stronger version of graceful labeling. [A graceful labeling f on a graph G is called an α -labeling, if there exists a positive integer λ such that for each edge uv of G either $f(u) \leq \lambda < f(v)$ or $f(v) \leq \lambda < f(u)$.]

In his classical paper, Rosa [6] proved the following decomposition result based on α -labeling.

Theorem 3 (Rosa [6]). *If a graph G with n edges has an α -labeling, then there exists a cyclic decomposition of the complete graph K_{2cn+1} into subgraphs isomorphic to G , where c is an arbitrary positive integer.*

Interesting part of α -labeling is that α -labeled graphs decompose complete graphs and complete bipartite graphs. In 1996, El-Zanati and Vanden Eynden have shown that α -labeled graphs also decompose the complete bipartite graphs.

Theorem 4 (El-Zanati and Vanden Eynden [1]). *If a graph G with q edges has an α -labeling, then there exists a G -decomposition of the complete bipartite graph $K_{mq,nq}$, where m and n are arbitrary*

positive integers.

2 Generating Graceful Graphs by Vertex Switching

In this section, we prove that the graph obtained by switching any one of the vertices of any one of the K_2 in the non-graceful graph $\bar{K}_n + 2K_2$, for $n \geq 2$ is graceful.

Theorem 5. *The graph obtained by switching any one of the vertices of any one of the K_2 in the non-graceful graph $\bar{K}_n + 2K_2$, for $n \geq 2$ is graceful.*

Proof. Let $u_1, u_2, \dots, u_n$ denote the vertices of $\bar{K}_n$ and let (u, v) , (x, y) denote the two copies of K_2 in the graph $\bar{K}_n + 2K_2$. Consider the graph obtained by switching any one of the vertices say u of K_2 in the graph $\bar{K}_n + 2K_2$, $(\bar{K}_n + 2K_2)_{\bar{u}}$. For the convenience, we denote this graph as G .

By the definition of vertex switching, u is adjacent to exactly two vertices x and y in G .

By the definition of G , $|V(G)| = n + 4$ and $|E(G)| = 3n + 3$.

Let $M = |E(G)| = 3n + 3$.

Then define $f : V(G) \rightarrow \{0, 1, 2, \dots, M\}$ in the following way.

$f(x) = 0$, $f(u) = n + 2$, $f(v) = n + 1$, $f(u_1) = M$.

For $2 \leq i \leq n$, define $f(u_i) = f(u_{i-1}) - 1$ and $f(y) = f(u_n) - 1$.

From the definition of f on $V(G)$, if the labels of vertices of G are arranged as a sequence, $f(u_1), f(u_2), \dots, f(u_n), f(y), f(u), f(v), f(x)$, then it forms a monotonically decreasing sequence.

Hence the vertex labels of the graph G are distinct.

Let $A_1 = \{u_1x, u_2x, \dots, u_nx, xy\}$, $A_2 = \{u_1v, u_2v, \dots, u_nv\}$, $A_3 = \{xu, yu\}$ and $A_4 = \{u_1y, u_2y, \dots, u_ny\}$. We give below the edge labels of the edges in the sets A_1, A_2, A_3 and A_4 respectively and we denote these sets by A'_1, A'_2, A'_3 and A'_4 respectively.

$A'_1 = \{M, M - 1, M - 2, \dots, M - n\}$,

$A'_2 = \{M - n - 1, M - n - 2, \dots, M - 2n\}$,

$A'_3 = \{M - 2n - 1, M - 2n - 2\}$ and

$A'_4 = \{M - 2n - 3, M - 2n - 4, \dots, 2, 1\}$.

The labels in the above sets can be arranged as a monotonically decreasing sequence and it ranges from M to 1.

Thus, the edge labels of G are distinct.

Hence, the graph G admits graceful labeling. $\square$

3 α -labeling of t -levels Shadow Graph of α -labeled Graphs

Lemma 6. *Let G be a graph with n vertices and m edges. Then, for $t \geq 2$, the t -levels shadow graph of G , $S_t(G)$ has tn vertices and $(3t - 2)m$ edges.*

Proof. Let G be a graph with n vertices and m edges. Let $V = \{v_1, v_2, \dots, v_n\}$ denote the vertex set of G . Consider t copies, $G_1, G_2, \dots, G_t$ of G . Let $V_i = \{v_{i1}, v_{i2}, \dots, v_{in}\}$ be the vertex set of the i^{th} copy G_i of G . Then the t -levels shadow graph of G , $S_t(G)$ has the vertex set $W = \bigcup_{i=1}^n V_i$.

From the definition of $S_t(G)$, we observe that

- (i) for each i , $1 \leq i \leq t$ two vertices v_{ij} and v_{ik} in G_i are adjacent if and only if v_j and v_k are adjacent in G ,
- (ii) further, if v_j is adjacent to $v_{j_1}, v_{j_2}, \dots, v_{deg\ v_j}$ in G , then for every j , $1 \leq j \leq n$, v_{ij} in G_i is adjacent to precisely $v_{(i+1)j_1}, v_{(i+1)j_2}, \dots, v_{(i+1)j_{deg\ v_j}}$, (the copies of the adjacent vertices of v_j of G in G_{i+1}) in G_{i+1} .

Then it is clear that $|V(S_t(G))| = t|V(G)|$. It follows from the

definition of $S_t(G)$, $|E(S_t(G))| = tm + \sum_{i=1}^{t-1} |E_{i(i+1)}|$, where $E_{i(i+1)}$

denotes the set of edges of $S_t(G)$ connecting the vertices of G_i and the vertices of G_{i+1} .

Let d_{ij} denote the degree of v_{ij} belonging to $V(G_i)$, for $1 \leq i \leq t$ and $1 \leq j \leq n$.

Then, $|E_{i(i+1)}| = \sum_{j=1}^n d_{ij} = 2|E(G_i)|$.

$$\begin{aligned} \text{Hence } |E(S_t(G))| &= tm + \sum_{i=1}^{t-1} 2|E(G_i)| = tm + 2(t-1)m \\ &= (3t-2)m. \end{aligned}$$

□

Theorem 7. For $t \geq 2$, the t -levels shadow graph of an α -labeled graph admits α -labeling.

Proof. Let G be an α -labeled graph with n vertices and m edges. Consider the t -levels shadow graph of G , $S_t(G)$ obtained by taking $t \geq 2$ copies of G and joining each vertex v_{ij} in G_i to the copies of its adjacent vertices in G_{i+1} , for $1 \leq j \leq n$ and $1 \leq i \leq t$, where $n = |V(G)|$.

By Lemma 6, $S_t(G)$ has tn vertices and $(3t-2)m$ edges.

Let $M = |E(S_t(G))| = (3t-2)m$.

Since G admits α -labeling, there exists an α -labeling

$f : V(G) \rightarrow \{0, 1, 2, \dots, m\}$ and a positive integer λ such that for every edge uv in G , either $f(u) \leq \lambda < f(v)$ or

$f(v) \leq \lambda < f(u)$. Hence the vertex set of the graph G , $V(G)$ can be partitioned into two sets say $V_1 = \{x \in V(G) / f(x) \leq \lambda\}$ and $V_2 = \{x \in V(G) / f(x) > \lambda\}$.

Since G_i is the i^{th} copy of G , as G is bipartite, G_i is also bipartite, for $1 \leq i \leq t$.

Hence, the vertex set of G_i can be partitioned into two sets as V_{i1}, V_{i2} , for $i, 1 \leq i \leq t$.

For the convenience, we describe the vertices of V_1 as $v_1, v_2, \dots, v_\ell$ and the vertices of V_2 as $v_{\ell+1}, v_{\ell+2}, \dots, v_n$. Also for the convenience, we describe the vertices of V_{i1} as $v_{i1}, v_{i2}, \dots, v_{i\ell}$ and the vertices of V_{i2} as $v_{i(\ell+1)}, v_{i(\ell+2)}, \dots, v_{in}$.

Define $g : V(S_t(G)) \rightarrow \{0, 1, 2, \dots, M\}$ in the following way.

$$g(v_{1j}) = \begin{cases} f(v_j), & \text{if } 1 \leq j \leq \ell, \\ f(v_j) + 3(t-1)m, & \text{if } \ell+1 \leq j \leq n, \end{cases}$$

$$\text{for } 2 \leq i \leq t, g(v_{ij}) = \begin{cases} g(v_{(i-1)j}) + 2m, & \text{if } 1 \leq j \leq \ell, \\ g(v_{(i-1)j}) - m, & \text{if } \ell+1 \leq j \leq n \end{cases}$$

From the definition of g defined on $V(S_t(G))$, if the labels of vertices of $S_t(G)$ are arranged as a sequence,

$$\begin{aligned} &g(v_{11}), g(v_{12}), \dots, g(v_{1\ell}), g(v_{21}), g(v_{22}), \dots, g(v_{2\ell}), \dots, \\ &g(v_{t1}), g(v_{t2}), \dots, g(v_{t\ell}), g(v_{t(\ell+1)}), g(v_{t(\ell+2)}), \dots, g(v_{tn}), \\ &g(v_{(t-1)(\ell+1)}), g(v_{(t-1)(\ell+2)}), \dots, g(v_{(t-1)n}), \dots, \end{aligned}$$

$g(v_{1(\ell+1)}), g(v_{1(\ell+2)}), \dots, g(v_{1n})$, then it forms a monotonically increasing sequence.

From the definition of g on $V(S_t(G))$, there exists an integer $\lambda + 2(t-1)m$ such that for each edge uv in $S_t(G)$,

$$f(u) \leq \lambda + 2(t-1)m < f(v).$$

The set of the labels of the edges of $S_t(G)$ can be partitioned into $2t-1$ sets having the labels of the edges of G_1 , followed by the labels of the edges joining the vertices of G_1 and G_2 , followed by the labels of the edges of G_2 and so on. Thus, the labels of the edges of $S_t(G)$ can be arranged as $\{M, M-1, \dots, (3t-3)m+1\} \cup \{(3t-3)m, (3t-3)m-1, \dots, (3t-4)m+1\} \cup \{(3t-4)m, (3t-4)m-1, \dots, (3t-5)m+1\} \cup \dots \cup \{m, m-1, \dots, 2, 1\}$. From these arrangements of the labels of the edges of $S_t(G)$, we observe that the labels of the edges of $S_t(G)$ forms a monotonically decreasing sequence. Thus, the labels of the edges of $S_t(G)$ are distinct and it ranges from M to 1.

Therefore, the graph $S_t(G)$ admits α -labeling. $\square$

The following corollary is an immediate consequence of Rosa's theorem [6].

Corollary 8. *For every t -levels shadow graph of an α -labeled graph G , $S_t(G)$, there exists a cyclic decomposition of the edges of the complete graph K_{2cq+1} into $(2cq+1)c$ copies of $S_t(G)$, where $q = |E(S_t(G))|$ and c is an arbitrary positive integer.*

The next corollary follows as an implication of El-Zanati et al's theorem [1].

Corollary 9. *For every t -levels shadow graph of an α -labeled graph G , $S_t(G)$, there exists a decomposition of the edges of the complete bipartite graph $K_{mq,nq}$ into mnq copies of $S_t(G)$, where $q = |E(S_t(G))|$ and m and n are arbitrary positive integers.*

Discussion

Theorem 5 illustrate that vertex switching in the non-graceful graph $\bar{K}_n + 2K_2$, for all $n \geq 2$ results in a graceful graph. Thus, it motivates to explore, what are the other graph operations that can bring gracefulness in the non-graceful graphs? Also, what are the other non-graceful graphs on which this operation can be applied so that the resultant graph is graceful?

Theorem 7 illustrates that t -levels shadow graph of an α -labeled graph admits α -labeling, the stronger version of graceful labeling. It motivates to ask the following questions.

- (i) What are the graphs H , for which we can find an α -labeled graph G , such that $S_t(G) = H$?
- (ii) What are the other graph operations when they are applied on graceful graphs or α -labeled graphs, the resultant graphs are also graceful or would admit α -labeling?

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