

OPEN GEODETIC NUMBER OF CERTAIN NETWORKS

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Abstract

For two vertices x and y of a connected graph G , a vertex $u \in V(G)$ is said to be geodominated by the pair $\{x, y\}$ if u lies on some $x - y$ geodesic in G . The geodetic interval $I[x, y]$ consists of x, y together with all vertices geodominated by the pair $\{x, y\}$. If S is a set of vertices of G , then the geodetic closure $I[S]$ is the union of all sets $I[x, y]$ for $x, y \in S$. If $I[S] = V(G)$, then S is said to be a geodetic set of G . The geodetic number $g(G)$ is the minimum cardinality of a geodetic set. A set S of vertices of a connected graph G is an open geodetic set of G if for each vertex v in G , either (i) v is a simplicial vertex of G and $v \in S$ or (ii) v is an internal vertex of a $x - y$ geodesic for some $x, y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the open geodetic number, denoted by $og(G)$. In this paper we determine the open geodetic number of Hexagonal mesh network, Apollonian network, Silicate network and few other interconnection networks.

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1 Introduction

The pioneering work of Harary, Chartrand and others concerning geodetic sets in graphs [1, 2, 3], made the study of geodetic related problems become a wide branch of study in graph theory. Let $G = (V, E)$ be a graph with vertex set V and edge set E . For two vertices x and y in a graph G , the distance $d(x, y)$ is the length of a shortest $x - y$ path in G .

A $x-y$ path of length $d(x, y)$ is called a $x-y$ geodesic. The closed interval $I[x, y]$ is the set of vertices of all $x-y$ geodesics of G . For $S \subseteq V(G)$, $I[S] = \bigcup_{x, y \in S} I[x, y]$. A set S of vertices of a graph G is a geodetic set if $I[S] = V(G)$, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. For an introduction to geodetic sets, see [6]. A set S of vertices of a connected graph G is an open geodetic set of G if for each vertex v in G , either (i) v is a simplicial vertex of G and $v \in S$ or (ii) v is an internal vertex of a $x-y$ geodesic for some $x, y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the open geodetic number denoted by $og(G)$.

2 Main Results

Definition 1. A vertex $u \in V(G)$ is said to be geodominated by the pair $\{x, y\}$ if u lies on some $x-y$ geodesic in G , for any $x, y \in V(G)$. The geodetic interval $I[x, y]$ consists of x, y together with all vertices geodominated by the pair $\{x, y\}$. If S is a set of vertices of G , then the geodetic closure $I[S]$ is the union of all sets $I[x, y]$ for $x, y \in S$. If $I[S] = V(G)$, then S is said to be a geodetic set of G . The geodetic number $g(G)$ is the minimum cardinality of a geodetic set.

Definition 2. A vertex v of a graph G is called simplicial if its neighborhood $N(v)$ induces a clique of size equal to degree of v . Simplicial vertices are also referred to as extreme vertices.

Definition 3. A set S of vertices in a connected graph G is an open geodetic set if for each vertex v in G , either (i) v is a simplicial vertex of G and $v \in S$ or (ii) v is an internal vertex of an $x-y$ geodesic, for some $x, y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the open geodetic number $og(G)$ of G .

Definition 4. [7] The Cartesian product $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(g, h)(g', h') \in E(G \square H)$ whenever either $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$.

Definition 5. [13] Let P_n denote a path on n vertices. For $m, n \geq 2$, $P_m \square P_n$ is defined as the two dimensional mesh with m rows and n columns. It is denoted by $M_{m \times n}$.

Definition 6. [13] Let C_n and P_n denote a cycle and a path on n vertices respectively. For $m, n \geq 2$, $C_m \square P_n$ is defined as the two

dimensional cylinder with m rows and n columns. It is denoted by $CY_{m \times n}$.

Theorem 7. [8] *Let G be a simple connected graph of order n . Then $2 \leq g(G) \leq og(G) \leq n$.*

Theorem 8. [8] *Every open geodetic set of a graph G contains its simplicial vertices. Also, if the set S of all simplicial vertices of G is an open geodetic set, then S is the unique minimum open geodetic set of G .*

Theorem 9. [5] *If a nontrivial connected graph G contains no simplicial vertices, then $og(G) \geq 4$*

Theorem 10. *For any graph G , $og(G \square H) \geq 4$*

Proof. Let $G \square H$ be the Cartesian product of graphs G and H . $G \square H$ contains no simplicial vertices. By Theorem 9, $og(G \square H) \geq 4$. $\square$

Theorem 11. $og(M_{m \times n}) = 4$, for $m, n \geq 2$.

Proof. For $1 \leq i \leq m$ and $1 \leq j \leq n$, label the i^{th} row and j^{th} column vertex of $M_{m \times n}$ as a_{ij} . Since $M_{m \times n}$ contains no simplicial vertices, by Theorem 9, $og(M_{m \times n}) \geq 4$. Let $S = \{a_{11}, a_{1n}, a_{m1}, a_{mn}\}$, that is, S contains the two degree vertices of $M_{m \times n}$. It is clear that every vertex of $M_{m \times n}$ lies in a geodesic between the vertices of S . Then S is an open geodetic set of G . Therefore $og(G) = 4$. $\square$

Theorem 12. $og(CY_{m \times n}) = 4$, where m and n are even.

Proof. The cylinder graph $CY_{m \times n}$ contains mn vertices. We follow the same labeling as in the previous theorem. Since $CY_{m \times n}$ does not contain any simplicial vertex, $og(CY_{m \times n}) \geq 4$. Let $S = \{a_{11}, a_{1(\frac{n}{2}+1)}, a_{m1}, a_{m(\frac{n}{2}+1)}\}$. Clearly all the vertices of $CY_{m \times n}$ lie in a geodesics between the vertices of S with $|S| = 4$. Therefore $og(CY_{m \times n}) = 4$, where m and n are even. $\square$

Definition 13. [9] Two vertices u and v of G are said to be antipodal if $d(u, v) = \text{diam } G$. Two edges uv and xy are antipodal if u is antipodal to x and v is antipodal to y .

Definition 14. [14] For $r \geq 2$, let Q_r denote the graph of the r -dimensional hypercube. $V(Q_r) = \{x_0x_1...x_{r-1} : x_i = 0 \text{ or } 1, 0 \leq i \leq r-1\}$. Two vertices $x_0x_1...x_{r-1}$ and $y_0y_1...y_{r-1}$ are adjacent if and only if they differ exactly in one position.

Theorem 15. For $r \geq 2$, $og(Q_r) = 4$.

Proof. The Hypercube graph Q_r does not contain a simplicial vertex. Then $og(Q_r) \geq 4$. Now let S be the end vertices of antipodal edges uv and xy of Q_n . Then S is an open geodetic set. Therefore $og(Q_n) = 4$. $\square$

Theorem 16. *Let S be a geodetic set of a graph G . Then there exists an open geodetic set T of G such that $S \subseteq T$.*

Proof. We have the following cases.

Case 1. S contains only simplicial vertices.

Since S is a set which contains only simplicial vertices then $T = S$ is an open geodetic set.

Case 2. S contains non-simplicial vertices.

For any non-simplicial vertex $x \in S$, we can find a $u - v$ geodesic such that x lies in a $u - v$ geodesic, where $u \in S$ and $v \in V \setminus S$. Proceeding this way with every non-simplicial vertices in S , we obtain the set T as the union of all such vertices u and v . It is straightforward that $S \subseteq T$ and T is the open geodetic set of G . $\square$

3 Split graph

A split graph is one whose vertex set can be partitioned as the disjoint union of an independent set and a clique. A graph G is split if and only if it does not have an induced subgraph isomorphic to one of the three forbidden graphs C_4 , C_5 , or $2K_2$. It follows from the definition that the complement, and every induced subgraph of a split graph is split.

Theorem 17. *Let G be a connected split graph. Then $og(G)$ is the set of all simplicial vertices of G .*

Proof. The vertex set $V(G)$ of a split graph G is the union of two sets $V_1(G)$ and $V_2(G)$ such that $V_1(G)$ contains the vertices of the clique whereas $V_2(G)$ contains the vertices of the maximal independent set of G . Let S be an open geodetic set of G . Then $S = A \cup V_2$ where $A \subset V_1$ contains all simplicial vertices of $G[V_1]$. The non-simplicial vertices from V_1 are openly geodominated either by the geodesics between the vertices of V_2 or by the geodesics between the vertices of V_2 and the vertices of V_1 or by both. $\square$

4 Hexagonal Mesh Networks

A Hexagonal Mesh Network is the triangular tessellation of hexagons. The n -dimensional Hexagonal Mesh is a 6-regular graph and a multi-processor interconnection network. An n -dimensional hexagonal mesh

denoted by $HX(n)$ has $3n^2 - 3n + 1$ vertices and $9n^2 - 15n + 6$ edges where n is the number of vertices on one side of the hexagon. The diameter of Hexagonal Mesh Network is $2(n-1)$. Its properties have been deeply studied and discussed in [10]. We also compute the open geodetic set of the Hexagonal Mesh Pyramid network denoted by $HXP(n)$. The number of vertices of $HXP(n)$ is n^3 and the number of edges of $HXP(n)$ is $3n^2(n-1) + 6(n-1)^3$. The diameter of Hexagonal Mesh Pyramid Network is $2(n-1)$. Hexagonal Mesh Pyramid Network is built with Hexagonal Mesh Network as the base. This section deals with the open geodetic sets of the Hexagonal Mesh Networks.

Theorem 18. *Let G be a Hexagonal Mesh Network. Then $og(HX) = 6$.*

Proof. Let S be the set of degree 3 vertices. That is $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ as shown in figure 1. Without loss of generality assume that $S \setminus v_6$ be the open geodetic set. Then neither the vertices v_1 and v_5 are simplicial nor it lie on a geodesic between the vertices in S . Therefore $og(HX) \geq 6$. And S clearly openly geodominates all the vertices of G . Hence $og(HX) \leq 6$. Therefore $og(HX) = 6$. $\square$

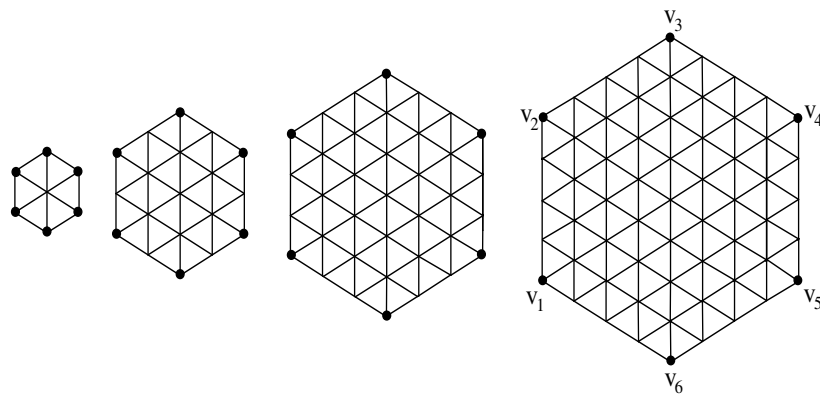


Figure 1

Theorem 19. *Let G be a Hexagonal Mesh Pyramid HXP_n of level n . Then $og(HXP_n) = 6$.*

Proof. Let S be the set of degree 4 vertices. That is $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ as shown in figure 2. Without loss of generality assume that $S \setminus v_6$ be the open geodetic set. Then neither the vertices v_1 and v_5 are simplicial nor it lie on a geodesic between the vertices in S . Therefore

$og(HXP_n) \geq 6$. And S clearly openly geodominates all the vertices of G . Hence $og(HXP_n) \leq 6$. Therefore $og(HXP_n) = 6$. $\square$

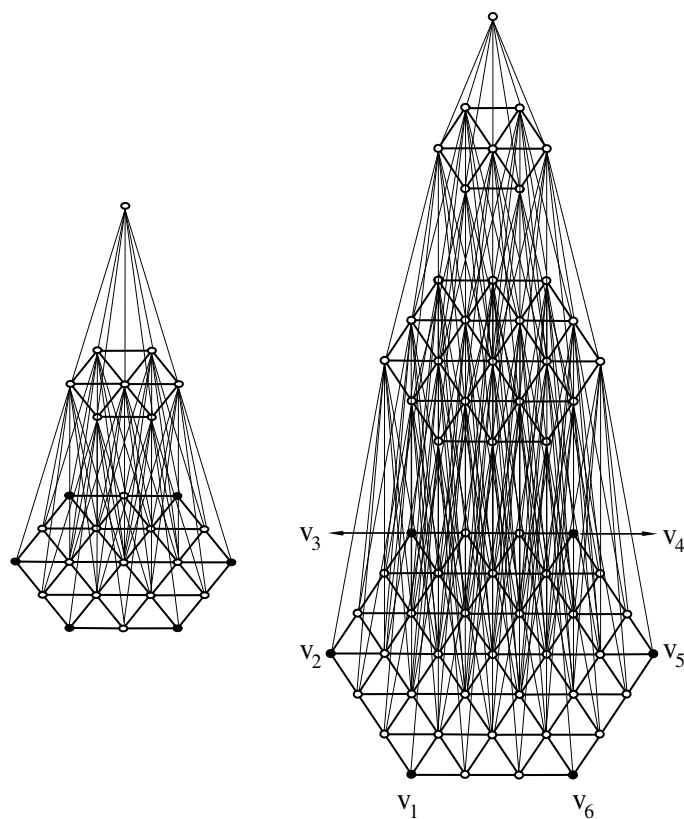


Figure 2

5 Apollonian Networks

Definition 20. An Apollonian network $A(n)$, $n \geq 0$, is a graph constructed as follows:

For $n = 0$, $A(0)$ is the complete graph K_3 .

For $n \geq 1$, $A(n)$ is obtained from $A(n-1)$: For each of the existing subgraphs of $A(n-1)$ that is isomorphic to K_3 and created at step $n-1$, a new vertex is introduced and connected to all the vertices of this subgraph. Figure 3 shows this construction process. The simplicial

vertices are in black color whereas the non-simplicial vertices are in white color.

It has $\frac{1}{2}(3^n + 5)$ vertices and $\frac{3}{2}(3^n + 1)$ edges, for $n \geq 1$.

Theorem 21. *Let $A(r)$ be the complete Apollonian network of level r . Then $og(A(r))$ is the set of all simplicial vertices.*

Proof. As in Figure 3, the black vertices are simplicial vertices and we denote the set of all simplicial vertices as S . Then it is clear that $og(A(r)) \geq |S|$. The complete Apollonian network of level r has 3^{r-1} simplicial vertices, for $r \geq 2$, whereas $A(0)$ and $A(1)$ has 3 and 4 simplicial vertices respectively. Since S construct the geodetic set for $A(r)$. Then $og(A(r)) \leq |S|$. Therefore $og(A(r)) = |S|$. $\square$

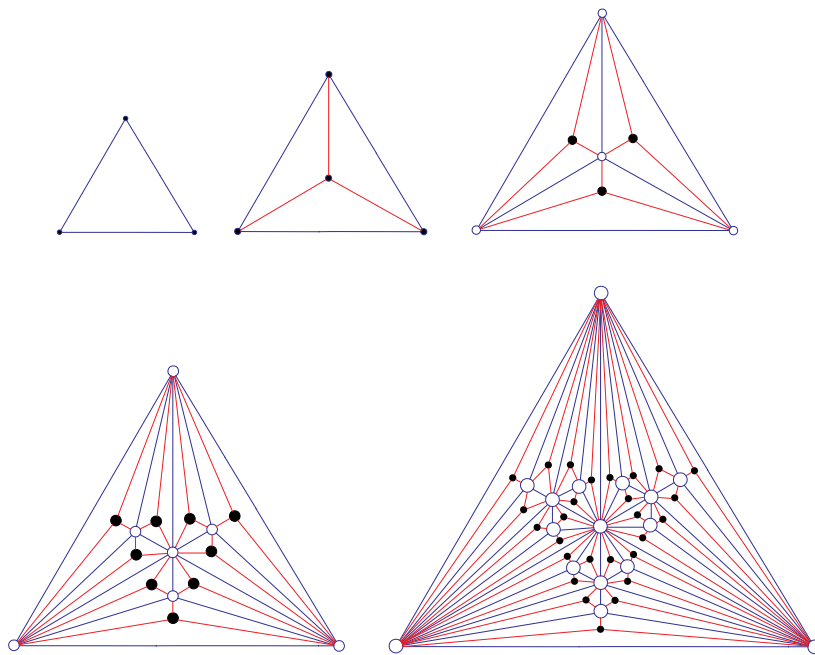


Figure 3: Complete Apollonian Networks

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