

Some Generating Functions of Generalized Hypergeometric 2D Polynomials by Lie Group Theoretic Method

P.L.Rama Kameswari^{1a,1b} and Dr V.S.Bhagavan²

^{1a}Department of Mathematics

K. L. University

Guntur Dt., A.P., India

^{1b}Department of Mathematics

Swarnandhra College of Engineering and Technology

Seetharampuram, Narsapuram-534 280

West Godawari Dt., A.P., India

ramaravikumar.i@gmail.com

²Department of Mathematics

K. L. University

Guntur Dt., A.P., India

drvsb002@kluniversity.in

Abstract

The object of this paper is to derive some bilinear / bilateral generating functions for the generalised hypergeometric 2D polynomials (*GH2DP*) $U_n(\beta; \gamma; x, y)$ with respect to y , by means of Lie group theoretic method. The generating functions are obtained by constructing a three dimensional Lie algebra isomorphic to a special linear group $SL(2, C)$ by giving a suitable interpretation to the index n of the polynomials $U_n(\beta; \gamma; x, y)$. Further, we have derived the well-known generating functions of Laguerre polynomial of two

variables as its application.

AMS Subject Classification: 33C45, 33C80, 33C05.

Key Words and Phrases: Generalized hypergeometric 2D polynomials, Special Linear Group, Generating functions.

1. Introduction

Generating relations of special functions play a very important role in investigation of various useful properties of the sequences which they generate. There are various methods of obtaining generating functions for variety of special functions. In present paper, we utilize group theoretic method of obtain the generating relations for generalised hypergeometric 2D polynomials $U_n(\beta; \gamma; x, y)$.

Group theoretic method proposed by Louis Weisner in 1955 and he employed this method to find generating relations for a large class of special functions. McBride[1], Miller[4], Srivastava and Manocha[9] reported group theoretic method for obtaining generating relations in their books. Khanna and Bhagavan[3] studied representation of a Lie algebra for the set of hypergeometric functions $\psi_{\alpha, \beta, \gamma, m}(x)$. Also Shukla and Meher[8] have done some works on Laguerre polynomials of two variables $L_n^{\alpha-n}(x, y)$ by using group theoretic method. Recently, Bhagavan V.S. and Rama Kameswari P.L. defined and studied generalized hypergeometric 2D polynomials (GH2DP), with the help of the following descending and ascending recurrence relations, respectively:

$$DU_n(\beta; \gamma; x, y) = nU_{n-1}(\beta; \gamma; x, y) \dots (1.1)$$

$$DU_n(\beta; \gamma; x, y) = \frac{1}{y(x-y)} \{(\gamma+n)U_{n+1}(\beta; \gamma; x, y) + [(n+\beta)x - (\gamma+2n)y]U_n(\beta; \gamma; x, y)\} \dots (1.2)$$

These two independent differential recurrence relations determine the second order linear ordinary differential equation

$$y(x-y)D^2U_n(x, y) - [(n+\beta-1)x - (\gamma+2n-2)y]DU_n(x, y) - n(\gamma+n-1)U_n(x, y) = 0 \dots (1.3)$$

where $D = \frac{d}{dy}$.

2. Representation of $SL(2, C)$ and Generating Functions

Let $SL(2, C)$ is the Lie algebra of a 3-dimensional complex local Lie group $SL(2, C)$.

$$SL(2, C) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d, \det g = 1 \right\} \text{--- (2.1)}$$

with commutation relations

$$[j^3, j^+] = j^+, \quad [j^3, j^-] = -j^-, \quad [j^+, j^-] = 2j^3 \dots (2.2)$$

In order to use Lie group theoretic method, we now construct the following partial differential equation by replacing $\frac{d}{dy}$ by $\frac{\partial}{\partial y}$, n by

$z \frac{\partial}{\partial z}$ and $U_n(\beta; \gamma; x, y)$ by $f(x, y, z)$ in (1.3) then the differential equation becomes

$$\left[y(x-y) \frac{\partial^2}{\partial y^2} + (2y-x)z \frac{\partial^2}{\partial y \partial z} - z^2 \frac{\partial^2}{\partial z^2} + \{(\gamma-2)y + (1-\beta)x\} \frac{\partial}{\partial y} - \gamma z \frac{\partial}{\partial z} \right] f(x, y, z) = 0 \dots (2.3)$$

Where $f(x, y, z) = z^n U_n(\beta; \gamma; x, y)$ is a solution of (2.3)

Let L represent the partial differential operator of (2.3), i.e.

$$L \equiv L \left(y, \frac{\partial}{\partial y}, z \frac{\partial}{\partial z} \right) = \left[y(x-y) \frac{\partial^2}{\partial y^2} + (2y-x)z \frac{\partial^2}{\partial y \partial z} - z^2 \frac{\partial^2}{\partial z^2} + \{(\gamma-2)y + (1-\beta)x\} \frac{\partial}{\partial y} - \gamma z \frac{\partial}{\partial z} \right] \dots (2.4)$$

Now we introduce the first order linearly independent differential operators J^3, J^+, J^- each of the form $A_1(x, y, z) \frac{\partial}{\partial y} + A_2(x, y, z) \frac{\partial}{\partial z} + A_3(x, y, z)$, Such that

$$J^3 = z \frac{\partial}{\partial z} + \frac{\gamma}{2}.$$

$$J^+ = zy(x-y) \frac{\partial}{\partial y} + z^2(2y-x) \frac{\partial}{\partial z} + z(\gamma y -)$$

$$J^- = -z^{-1} \frac{\partial}{\partial y} \dots (2.6)$$

These operators J^3, J^+, J^- satisfy the commutation relations

$$[J^3, J^\pm] = \pm J^\pm, \quad [J^+, J^-] = 2J^3 \dots (2.7)$$

It includes that the J -operators generates a three dimensional Lie algebra isomorphic to $SL(2C)$, which is algebra of generalized Lie derivatives of a multiplier representation $SL(2C)$.

In order to determine the multiplier representation of $SL(2C)$, we first compute the actions of $\exp(a'J^+)f$, $\exp(b'J^-)f$, and $\exp(c'J^3)f$ on $f \in F$, where F is the complex vector space of all functions of x, y and z analytic in some neighbourhood of the point $(x_0, y_0, z_0) =$

$(0, 0, 0)$.

If $g \in SL(2, C)$ has coordinates (a', b', c') we have $g = \exp(a' J^+) \exp(b' J^-) \exp(c' J^3)$, from which we get the operator $T(g)$ acting on f given by

$$[T(g)f](x, y, z) = (1 - a'yz)^{\beta-\gamma} \{1 + a'z(x-y)\}^{-\beta} \exp\left(\frac{\gamma c'}{2}\right) \\ f\left(x, (1 + a'z(x-y))\left(\frac{(1 + a'b')yz - b'}{z}, \frac{\exp(c')z}{(1 + a'z(x-y))(1 - a'yz)}\right)\right) \dots\dots(2.8)$$

which after setting $a' = -\frac{b}{d}, b' = -cd, \exp\left(\frac{c'}{2}\right) = \frac{1}{d}$ and using the fact that $ad - bc = 1$ gives

$$[T(g)f](x, y, z) = (d + bzy)^{\beta-\gamma} (d - bz(x-y))^{-\beta} f(x, \zeta, \eta) \dots\dots\dots(2.9)$$

where

$$\zeta = \frac{\{(d - bz(x-y))(azy + c)\}}{z}, \quad \eta = \frac{z}{\{(d - bz(x-y))(bzy + d)\}}$$

$$\left|\frac{bzy}{d}\right| < 1, \quad \left|\frac{c}{azy}\right| < 1, \quad \left|\frac{bz(x-y)}{d}\right| < 1.$$

For $f \in F$ and g lies in a small sufficiently neighbourhood of the identity element $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in SL(2, C)$.

In terms of J -operators, we introduce the Casimir operator[4] as

$$C = -[y(x-y)\frac{\partial^2}{\partial y^2} + (2y-x)z\frac{\partial^2}{\partial y\partial z} - z^2\frac{\partial^2}{\partial z^2} + \{(\gamma-2)y + (1-\beta)x\}\frac{\partial}{\partial y} - \gamma z\frac{\partial}{\partial z}] + \frac{\gamma(\gamma-2)}{4} \dots(2.10)$$

It can be verified that C commutes with J^+, J^- and J^3 .

Equation (2.10) enables us to write(2.3) as

$$Cf(x, y, z) = \frac{\gamma}{4}(\gamma-2)f(x, y, z) \dots\dots\dots(2.11)$$

To accomplish our job of obtaining generating functions, we search for the function $f(x, y, z)$ which satisfy (2.11)

Now we consider the following different cases:

Case 1. When $f(x, y, z)$ is a common eigen function of C and J^3 .

The simultaneous equations

$$Cf(x, y, z) = \frac{\gamma}{4}(\gamma-2)f(x, y, z) \dots\dots\dots(2.12)$$

$$J^3 f(x, y, z) = \left(\frac{\gamma}{4} + \nu\right) f(x, y, z) \dots \dots \dots (2.13)$$

Admit a solution that $f(x, y, z) = z^\nu U_\nu(\beta; \gamma; x, y)$. so that

$$[T(g)f](x, y, z) = (d + bzy)^{\beta-\gamma-\nu} (d - bz(x-y))^{-\beta-\nu} z^\nu U_\nu\left(\beta; \gamma; x, \frac{(d - bz(x-y))(azy + c)}{z}\right),$$

\

.....(2.14)

which satisfy the relation

$$C[T(g)f](x, y, z) = \frac{1}{4}\gamma(\gamma - 2)[T(g)f](x, y, z)$$

. If ν is not an integer, equation(2.14) has a laurent series expansion[9]:

$$[T(g)f](x, y, z) = \sum_{k=-\infty}^{\infty} P_k(g, x, y) z^{k+\nu} \dots \dots \dots (2.15)$$

If $[T(g)f](x, y, z)$ is regular at $x = 0$ and $y = 1$, we have,

$$[T(g)f](x, y, z) = \sum_{k=-\infty}^{\infty} A_{k\nu}(g) U_{k+\nu}(\beta; \gamma; x, y) z^{k+\nu} \dots \dots (2.16)$$

To determine the $A_{k\nu}(g)$, put $x = 0$ and $y = 1$ and equate the coefficient of z^k , we get

$$A_{k\nu}(g) = (-1)^k \frac{a^\nu b^k d^{-(\nu+k+\gamma)\Gamma(\gamma+\nu+k)}}{\Gamma(\nu+k)\Gamma(1+k)} {}_2F_1\left[\begin{matrix} -\nu, \nu + \gamma + k; & \frac{bc}{ad} \\ 1 + k; \end{matrix}\right] \dots \dots (2.17)$$

Thus the generating function (2.15) becomes

$$\begin{aligned} & (d + bzy)^{\beta-\gamma-\nu} (d - bz(x-y))^{-\beta-\nu} z^\nu U_\nu\left(\beta; \gamma; x, \frac{(d - bz(x-y))(azy + c)}{z}\right) \\ &= \sum_{k=-\infty}^{\infty} (-1)^k \frac{a^\nu b^k d^{-(\nu+k+\gamma)\Gamma(\gamma+\nu+k)}}{\Gamma(\nu+k)\Gamma(1+k)} {}_2F_1\left[\begin{matrix} -\nu, \nu + \gamma + k; & \frac{bc}{ad} \\ 1 + k; \end{matrix}\right] U_{\nu+k}(\beta; \gamma; x, y) z^{\nu+k} \dots \dots \dots (2.18) \end{aligned}$$

Where $|y| < 1, \left|\frac{bzy}{d}\right| < 1, \left|\frac{bz(x-y)}{d}\right| < 1$.

Deductions:

(1) $a = d = z = 1$ and $c = 0$ then equation(2.18) gives us

$$(1 - by)^{\beta-\gamma-\nu} \{1 - b(x-y)\}^{-\beta-\nu} U_\nu(\beta; \gamma; x, y(1 + b(x-y))) =$$

$$\sum_{k=0}^{\infty} \frac{(\gamma + \nu)_k}{k!} U_{\nu+k}(\beta; \gamma; x, y) b^\nu, |by| < 1, |b(x-y)| < 1 \dots \dots \dots (2.19)$$

(2) $a = d = z = 1$ and $b = 0$, then equation (2.18) gives us

$$U_\nu(\beta; \gamma; x, y - c) = \sum_{k=0}^{\infty} \frac{(-\nu)_k}{k!} U_{\nu-k}(\beta; \gamma; x, y) c^\nu \dots\dots\dots (2.20)$$

Case 2: when $f(x, y, z)$ is a common eigenfunction of C and J^- .
Let $f(x, y, z)$ be a solution of the simultaneous equations

$$Cf(x, y, z) = \frac{\gamma}{4}(\gamma - 2)f(x, y, z) \dots\dots\dots (2.21)$$

$$J^- f(x, y, z) = -f(x, y, z) \dots\dots\dots (2.22)$$

Assuming the general solution of (2.21) and (2.22) in the form

$$f(x, y, z) = e^{yz} K(z) \dots\dots\dots (2.23)$$

by this we get

$$\left(z \frac{\partial^2}{\partial z^2} + (\gamma + z) \frac{\partial}{\partial z} + \beta \right) K(z) = 0 \dots\dots (2.24)$$

This is Kummer's differential equation (9) and has for its solution

$$K(z) = {}_1F_1[\beta; \gamma; -xz] \dots\dots\dots (2.25)$$

Thus one solution of this system is

$$f(x, y, z) = e^{yz} {}_1F_1[\beta; \gamma; -xz] \dots\dots\dots (2.26)$$

From this, we have

$$[T(g)f](x, y, z) = (d + bzy)^{\beta-\gamma} (d - bz(x-y))^{-\beta} \exp\left(\frac{azy + c}{bzy + d}\right) {}_1F_1\left(\beta; \gamma; \frac{-xz}{(d - bz(x-y))(d + bzy)}\right) \dots\dots (2.27)$$

satisfying the relation

$$C[T(g)f](x, y, z) = \frac{1}{4}\gamma(\gamma - 2)[T(g)f](x, y, z) \dots\dots\dots (2.28)$$

Since $[T(g)f](x, y, z)$ is analytic at $z = 0$, it can be expanded in the form

$$[T(g)f](x, y, z) = \sum_{n=0}^{\infty} P_n(g) U_n(\beta; \gamma; x, y) z^n \dots\dots\dots (2.29)$$

To compute the coefficients of $P_n(g)$, we put $x = 0$ and $y = 1$ in (2.29)

$$(bz + d)^{-\gamma} \exp\left(\frac{azy + c}{bzy + d}\right) = \sum_{n=0}^{\infty} P_n(g) z^n \dots\dots\dots (2.30)$$

Which may be written as

From the familiar generating function for Laguerre polynomials [5], we have

$$(1 - t)^{-1-\alpha} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{\alpha}(x) t^n \dots\dots\dots (2.31)$$

On comparing (2.30) with the well known generatin function (2.31), we find

$$P_n(g) = d^{-\gamma} \exp\left(\frac{-b}{d}\right)^n L_n^{(\gamma-1)}\left(\frac{1}{bd}\right) \dots\dots\dots (2.32)$$

Thus we have

$$\begin{aligned} & \left(1 + \left(\frac{bzy}{d}\right)\right)^{\beta-\gamma} \left(1 - \frac{bz(x-y)}{d}\right)^{-\beta} \exp\left(\frac{azy + c}{bzy + d}\right) {}_1F_1\left(\beta; \gamma; \frac{-xz}{(d - bz(x-y))(bzy + d)}\right) \\ &= \sum_{n=0}^{\infty} \exp\left(\frac{c}{d}\right) L_n^{\gamma-1} \frac{1}{bd} z^n U_n(\beta; \gamma; x, y) \left(-\frac{b}{d}\right)^n \dots\dots\dots (2.33) \end{aligned}$$

Deduction:

By setting $a = i\sqrt{w}$, $b = \frac{i}{\sqrt{w}}$, $c = 0$, $d = -\frac{i}{\sqrt{w}}$, $i = \sqrt{-1}$.

we have

$$\begin{aligned} & (1-yz)^{\beta-\gamma} (1+z(x-y))^{-\beta} \exp\left(\frac{wzy}{zy-1}\right) {}_1F_1\left(\beta; \gamma; \frac{-xz}{(1+z(x-y))(1-yz)}\right) \\ &= \sum_{n=0}^{\infty} U_n(\beta; \gamma; x, y) L_n^{\gamma-1}(w) z^n \cdot |yz| < 1 \dots\dots\dots (2.34) \end{aligned}$$

which is bilateral generating function involving the Laguerre polynomial.

Applications

From the relations (2.19) and (2.20), we can derive the following generating functions for lagaurre polynoials of two variables.

$$\begin{aligned} 1. & (1-yt)^{-n-\alpha-1} \exp\left(\frac{-tx}{(1-ty)}\right) L_n^{(\alpha)}\left(x, \frac{x}{1-ty}\right) = \sum_{l=0}^n \frac{(1+n)_l}{l!} L_{n+l}^{(\alpha)}(x) t^l. \\ 2. & (1-t)^n L_n^{(\alpha)}\left(x, \frac{1-t}{y}\right) = \sum_{l=0}^n \frac{(\alpha-n)_l}{l!} L_{n-1}^{(\alpha)}(x, y) t^l. \end{aligned}$$

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