

Optimal Node Ordering and Layout of Chord Graphs

Jessie Abraham

Department of Mathematics

Loyola College, Chennai, 600 034, INDIA

jessie.abrt@gmail.com

Abstract

In this paper we obtain a nested solution of the edge isoperimetric problem for the chord graph, which in turn is used in finding the minimum layout of embedding the chord graph into binomial tree and caterpillar graph.

AMS Subject Classification: 05C78, 05C85

Key Words and Phrases: chord graph, node ordering, layout, optimal set

1 Introduction

A simple and undirected graph $G = (V_G, E_G)$ is an ordered pair of finite sets, where the set V_G contains all nodes of G , and the set $E_G \subseteq V_G \times V_G$ is the set of all edges $e = (i, j)$ of G . For any simple, undirected graph $G = (V_G, E_G)$ on n nodes, the optimal node ordering problem is to find an order $\mathcal{O} = \langle v_1, v_2, \dots, v_n \rangle$ of its node set such that for any i , $1 \leq i \leq n$, the first i nodes in that order satisfies the edge isoperimetric problem stated below [4, 6].

EIP 1: Let $G = (V_G, E_G)$ and $S \subseteq V_G$. Let $\vartheta_G(S) = \{(u, v) \in E_G : u \in S \text{ \& } v \in S\}$ and for $1 \leq k \leq |V_G|$, let $\vartheta_G(k) = \min_{S \subseteq V_G, |S|=k} |\vartheta_G(S)|$. Then the problem is to find $S \subseteq V_G$ with $|S| = k$ such that $\vartheta_G(k) = |\vartheta_G(S)|$.

EIP 2: Let $G = (V_G, E_G)$ and $S \subseteq V_G$. Let $I_G(S) = \{(u, v) \in E_G : u \in S \text{ \& } v \in S\}$ and for $1 \leq k \leq |V_G|$, let $I_G(k) =$

$\min_{S \subseteq V_G, |S|=k} |I_G(S)|$. Then the problem is to find $S \subseteq V_G$ with $|S| = k$ such that $I_G(k) = |I_G(S)|$.

The subset S which satisfies both the aforementioned conditions is known as an optimal set.

A Graph embedding is an ordered pair $\prec f, P_f \succ$ of one-to-one functions that maps a guest graph $G = (V_G, E_G)$ into a host graph $H = (V_H, E_H)$, where f is a map from V_G to V_H and P_f assigns to each edge (u, v) of G , a shortest path $P_f(u, v)$ in H [5,11]. The edge congestion of an embedding $EC_{\prec f, P_f \succ}(e)$ is the maximum number of edges of the guest graph that are embedded on any single edge e of the host graph [9]. The layout [7] of an embedding $\prec f, P_f \succ$ from G to H is defined as $L_{\prec f, P_f \succ}(G, H) = \sum_{e \in E_H} EC_f(e)$. The layout of G into H is given by $L(G, H) = \min_{\prec f, P_f \succ} L_{\prec f, P_f \succ}(G, H)$. Finding the nested solution of edge isoperimetric problem can be used in solving layout problem for certain graphs [1–3, 8–10]. and has various real life applications [7].

Lemma 1. (*k-Partition Wirelength Lemma*) [3] Let G be an r -regular graph and $\prec f, P_f \succ$ be an embedding of G into a graph H . Let $E^k(H)$ denote a collection of edges of H with each edge in H repeated exactly k times. Let $\{S_1, S_2, \dots, S_p\}$ be a partition of $E^k(H)$ such that each S_i is an edge cut of H . For $1 \leq i \leq p$, the removal of edges of S_i splits H into 2 components H_i^1 and H_i^2 . Let $G_i^1 = G[f^{-1}(H_i^1)]$ and $G_i^2 = G[f^{-1}(H_i^2)]$. Let $EC_{\prec f, P_f \succ}$ denotes the sum of edge congestion over all the edges in S_i . S_i satisfies the following conditions for $EC_{\prec f, P_f \succ}$ to be minimum.

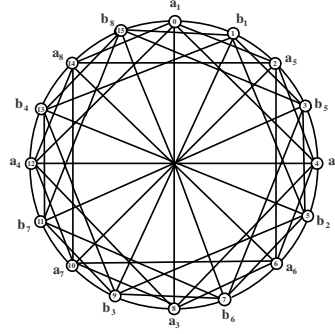
- (i) For every edge $(a, b) \in G_i^j$, $j = 1, 2$, $P_f(a, b)$ has no edges in S_i and for every edge $(a, b) \in G$ with $a \in G_i^1$ and $b \in G_i^2$, $P_f(a, b)$ has exactly one edge in S_i .

- (ii) Either $V(G_i^1)$ or $V(G_i^2)$ is an optimal node set.

Then, $EC_{\prec f, P_f \succ}(S_i) = \sum_{e \in S_i} EC_{\prec f, P_f \succ}(e) = r|V(G_i^1)| - 2|E(G_i^1)|$ and

$$L_{\prec f, P_f \succ}(G, H) = \frac{1}{k} \sum_{i=1}^p EC_{\prec f, P_f \succ}(S_i).$$

In this paper, we find the optimal node ordering of a chord graph and find the layout of chord graphs into binomial tree and caterpillar and prove its correctness using the k -Partition Wirelength Lemma.

Figure 1: Chord graph CH_4

2 Optimal Node Ordering of Chord Graphs

Definition 2. [10] A chord graph CH_n with 2^n nodes has node set $V(CH_n) = \{0, 1, \dots, 2^n - 1\}$. Edge $e = (i, j) \in E(CH_n)$ if and only if $i + 2^k = \text{mod}_{2^n} j$ or $j + 2^k = \text{mod}_{2^n} i$, for all $k \in \{0, 1, \dots, n-1\}$. See Figure 1. CH_n is $(2n - 1)$ regular.

Definition 3. [10] A subchord of CH_n is its subgraph induced by i , $1 \leq i \leq 2^n$ nodes. Two subchords CH_{k_1} and CH_{k_2} on k_1 and k_2 nodes respectively are said to be strong neighboring subchords, if each node of CH_{k_1} (or) CH_{k_2} has exactly two neighbors in CH_{k_2} (or) CH_{k_1} .

Definition 4. [10] Let $l = \sum_{j=1}^p 2^{c_j}$, $0 \leq c_1 < c_2 < \dots < c_p$ be the base 2 representation of for any positive integer $l < 2^n$. If $S \subseteq V_{CH_n}$ is a subset on l nodes made up of a disjoint union of c_j -subchords, $1 \leq j \leq p$, such that each c_i -subchord lies in the strong neighborhood of every c_j -subchord for $j > i$, then the subchord induced by S is called an IChord.

Remark 5. [10] If $CH_n[S]$ is an IChord and $|S|$ is even, then $|E_{CH_n[S]}| = \sum_{j=1}^p (p-1)2^{c_j+1} + 2^{c_j-1}(2c_j-1)$ and if $|S|$ is odd, then $|E_{CH_n[S]}| = \sum_{j=2}^p (p-1)2^{c_j+1} + 2^{c_j-1}(2c_j-1)$.

For $1 \leq l \leq 2^n$, if $S \subseteq V_{CH_n}$, $|S| = l$, is an IChord, then $I_G(l) = |I_G(S)|$. Since CH_n is a regular graph, if for a subset $S \subseteq V_{CH_n}$, $I_G(l) = |I_G(S)|$ then $\vartheta_G(l) = |\vartheta_G(S)|$ and hence S is an optimal set.

Let $a_1 = 0$, $a_2 = 2^{n-2}$, $a_3 = 2^{n-1}$, $a_4 = 2^{n-1} + 2^{n-2}$. For $1 \leq j \leq 4$ and $0 \leq r \leq 2^{n-3} - 1$, let $a_{j+4r} = a_j + 2r$. Also, let $b_1 = 1$, $b_2 = 2^{n-2} + 1$, $b_3 = 2^{n-1} + 1$, $b_4 = 2^{n-1} + 2^{n-2} + 1$. For $1 \leq j \leq 4$ and $0 \leq r \leq 2^{n-3} - 1$, let $b_{j+4r} = b_j + 2r$.

Lemma 6. For $1 \leq i \leq 2^{n-1}$, $A_i = \{a_k : 1 \leq k \leq i\}$ and $B_i = \{b_k : 1 \leq k \leq i\}$ are optimal sets in CH_n .

Proof. For $1 \leq i \leq 4$, the subchord CH_i formed by $V(CH_n[A_i])$ is a complete graph on i nodes and hence is optimal. For $i > 4$, let the nodes in A_i be grouped into sets featuring the base 2 representation of i , beginning with the pack containing the ultimate node in the set with lowest power and progressing backwards, ending with the first 2^j nodes in the set with the highest power j .

Clearly, every node in sets with lower power has a strong neighborhood in sets with higher power and hence $CH_n[A_i]$ forms an IChord.

Now, define a mapping π from $V(CH_n[A_i])$ to $V(CH_n[B_i])$ such that $\pi(x) = x + 1$. For any two nodes $x, y \in V(CH_n)$, $(x, y) \in E(CH_n[A_i]) \Leftrightarrow x + 2^k = \text{mod}_{2^n} y$ or $y + 2^k = \text{mod}_{2^n} x$, for all $k \in \{0, 1, \dots, n-1\} \Leftrightarrow (x+1) + 2^k = \text{mod}_{2^n} (y+1)$ or $(y+1) + 2^k = \text{mod}_{2^n} (x+1)$, for all $k \in \{0, 1, \dots, n-1\} \Leftrightarrow (x+1, y+1) \in E(CH_n[B_i])$. Hence π is an isomorphism and B_i is an optimal set in CH_n . $\square$

Theorem 7. For $1 \leq i \leq 2^n$,

$$(AB)_i = \begin{cases} A_i & : 1 \leq i \leq 2^{n-1} \\ A_{2^{n-1}} \cup B_{i-2^{n-1}} & : 2^{n-1} < i \leq 2^n \end{cases}$$

is an optimal set in CH_n .

Proof. The proof can be split into two parts in accordance with the value of i .

Part A ($i \leq 2^{n-1}$): Proof follows from Lemma 6.

Part B ($i > 2^{n-1}$): Both $CH_n[A_{2^{n-1}}]$ and $CH_n[B_{i-2^{n-1}}]$ are IChords as shown in Lemma 6. It is evident from the construction of CH_n

that each node in $CH_n[B_{i-2^{n-1}}]$ is adjacent to exactly two nodes in $CH_n[A_{2^{n-1}}]$. Hence the two subgraphs have a strong neighborhood, implying that their union forms an IChord. $\square$

Remark 8. By the construction of CH_n , we make an observation wherein, if the consecutive nodes of the optimal set $(AB)_{2^n}$ are decomposed into 2^{n-i} sets with each set carrying 2^i nodes, every one of these sets is isomorphic to $(AB)_{2^i}$ and hence optimal.

3 Layout of Embedding Chord Graph into Binomial Tree and Caterpillar

In this section we compute the layout of embedding the chord graph into trees such as binomial tree and caterpillar.

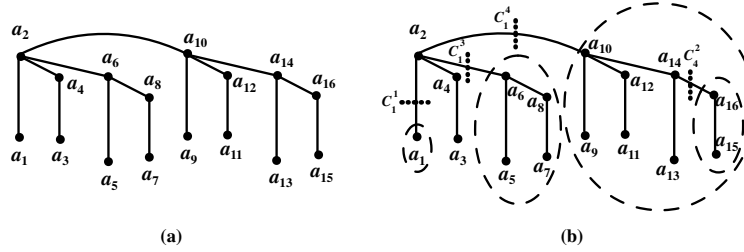
Definition 9. A binomial tree B_0 of height 0 is a single vertex. For $n > 0$, a binomial tree B_n of height n is a tree formed by joining the roots of two binomial trees of height $n - 1$ with a new edge and designating one of these roots to be the root of the new tree. A binomial tree of height n has 2^n nodes.

Table 1: Embedding labeling

Guest Graph	Host Graph	Embedding
Label a node as 0. Label the consecutive nodes as $x+1$ in the clockwise direction starting at $x = 0$ and ending at $x = 2^n - 1$. See Figure 1.	Label the nodes of B_n by inorder tree labeling [?] from a_1 to a_{2^n} . See Figure 2(a).	Define an embedding $\prec f, P_f \succ$ from CH_n into B_n as $f(x) = x$ together with $P_f(u, v)$ where $P_f(u, v)$ is a shortest path in B_n between $f(u)$ and $f(v)$ for $(u, v) \in E(CH_n)$.

Lemma 10. The embedding labeling in Table 1 induces a minimum layout.

Proof. For $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, 2^{n-j}$, let $C_i^j =$

Figure 2: (a) Binomial tree B_4 (b) Edge cuts of B_4

$\{(a_{2^{j-1}(2i-2)}, a_{2^{j-1}(2i-1)})\}$ be the edge cut of B_n such that C_i^j splits B_n into components X_i^j and $\bar{X}_i^j$ where $V_{X_i^j} = \{a_{2^{j-1}(2i-1)}, a_{2^{j-1}(2i-1)+1}, \dots, a_{2^{j-1}(2i-1)+(2^{j-1}-1)}\}$. See Figure 2(b). Let $(CH_n)_i^j$ be the graph induced by $\{f(v) : v \in V_{X_i^j}\}$ under $\prec, f, P_f \succ$. By Theorem 7 and Remark 8, $V_{(CH_n)_i^j}$ is an optimal set in CH_n . The edge cut C_i^j satisfies conditions (i) – (iii) of the 1-Partition Wirelength Lemma. Therefore, for $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, 2^{n-j}$, $EC_{\prec, f, P_f \succ}(C_i^j)$ is minimum and $L(CH_n, B_n) = L_{\prec, f, P_f \succ}(CH_n, B_n)$. $\square$

Theorem 11. The layout of embedding CH_n into B_n is given by $L(CH_n, B_n) = 2^{n-1}(n^2 + n - 1)$.

Proof. By Lemma 1 and Remark 5, for $j = 1, 2, \dots, n$ and $i = 1, \dots, 2^{n-j}$, $EC_{\prec, f, P_f \succ}(C_i^j) = (2n - 1)(2^{j-1}) - 2(2^{j-2}(2j - 3))$. Therefore $L(CH_n, B_n) = \sum_{j=1}^n \sum_{i=1}^{2^{n-j}} EC_{\prec, f, P_f \succ}(C_i^j) = \sum_{j=1}^n \sum_{i=1}^{2^{n-j}} \{(2n - 1)(2^{j-1}) - 2^{j-1}(2j - 3)\} = n2^{n-1}(2n - 1) - \sum_{j=2}^n \sum_{i=1}^{2^{n-j}} 2^{j-1}(2j - 3) = 2^{n-1}(n^2 + n - 1)$. $\square$

A tree is called a caterpillar if the deletion of all the leaf nodes gives a path. This path is called the spine of the caterpillar. Equivalently a caterpillar can be defined as follows.

Definition 12. [8] For $m \geq 1$ and $i = 1, 2, \dots, m$, let k_i be a collection of non-negative integers such that $m + k_1 + \dots + k_m \geq 3$. A tree which is obtained from a path $v_1, v_2, \dots, v_m$ by joining each v_i to a set of leaf nodes v_{ij} ($j = 1, 2, \dots, k_i$) is called a caterpillar

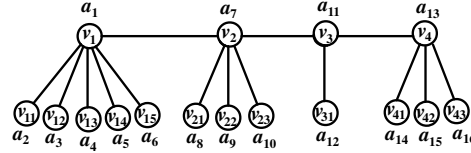


Figure 3: Caterpillar $CAT(5, 3, 1, 3)$ labeled in optimal order

and is denoted by $CAT(k_1, k_2, \dots, k_m)$. See Figure 3.

Theorem 13. [8] *The minimum layout of embedding an r -regular graph G with optimal order into $CAT(k_1, k_2, \dots, k_m)$ is given by $L(G, CAT(k_1, k_2, \dots, k_m)) = \sum_{i=1}^{m-1} \{r(k_1 + \dots + k_i + i) - 2|E(G[k_1 + \dots + k_i + i])|\} + r(k_1 + k_2 + \dots + k_m)$.*

Optimal Labeling: Label the nodes of the caterpillar graph $CAT(k_1, \dots, k_m)$ as follows: Label the node v_i as $a_{(k_1 + \dots + k_{i-1} + (i-1))}$ for $i = 1, 2, \dots, m$ and v_{ij} as $a_{(k_1 + \dots + k_{i-1} + (i-1) + j)}$ for $i = 1, 2, \dots, m$, $j = 1, 2, \dots, k_i$, where $a_{k_0} = 0$. See Figure 3.

As a consequence of Theorem 13, we have the following result.

Theorem 14. *The layout of embedding CH_n with optimal order $(AB)_{2^n}$ into $CAT(k_1, k_2, \dots, k_m)$ on 2^n nodes is given by $L(CH_n, CAT(k_1, k_2, \dots, k_m)) = \sum_{i=1}^{m-1} \{(2n-1)(k_1 + \dots + k_i + i) - 2|E(CH_n[(AB)_{k_1 + \dots + k_i + i}])|\} + (2n-1)(k_1 + k_2 + \dots + k_m)$.*

4 Conclusion

In this paper, we have obtained an optimal ordering of the node set of a chord graph satisfying the combinatorial edge isoperimetric problem and used the optimal set to compute the layout of embedding chord graph into binomial tree and caterpillar graphs.

References

- [1] J. Abraham, M. Arockiaraj, Wirelength of enhanced hypercubes into r -rooted complete binary trees, *Electron. Notes Discrete Math.* **53** (2016), 373–382.
- [2] M. Arockiaraj, J. Abraham, J. Quadras, A.J. Shalini, Linear layout of locally twisted cubes, *Int. J. Comp. Math.* **94**(1) (2017), 56–65.
- [3] M. Arockiaraj, J. Quadras, I. Rajasingh, A.J. Shalini, Embedding hypercubes and folded hypercubes onto Cartesian product of certain trees, *Discrete Optim.* **17** (2015), 1–13.
- [4] S.L. Bezrukov, S.K. Das, R. Elsässer, An edge-isoperimetric problem for powers of the Petersen graph, *Ann. Comb.* **4**(2) (2000), 153–169.
- [5] S.L. Bezrukov, B. Monien, W. Unger, G. Wechsung, Embedding ladders and caterpillars into the hypercube, *Discrete Appl. Math.* **83**(1-3) (1998), 21–29.
- [6] M.R. Garey, D.S. Johnson, *Computers and Intractability, A Guide to the Theory of NP-Completeness*, Freeman, San Francisco (1979).
- [7] L.H. Harper, *Global Methods for Combinatorial Isoperimetric Problems*, Cambridge University Press, Cambridge, UK (2004).
- [8] P. Manuel, M. Arockiaraj, I. Rajasingh, B. Rajan, Embedding hypercubes into cylinders, snakes and caterpillars for minimizing wirelength, *Discrete Appl. Math.* **159**(17) (2011), 2109–2116.
- [9] I. Rajasingh, M. Arockiaraj, Linear wirelength of folded hypercubes, *Math. Comput. Sci.* **5**(1) (2011), 101–111.
- [10] H. Rostami, J. Habibi, Minimum linear arrangement of chord graphs, *Appl. Math. Comput.* **203** (2008), 358–367.
- [11] J.-M. Xu, *Topological Structure and Analysis of Interconnection Networks*, Kluwer Academic Publishers (2001).

