

**Delay-dependent criterion for finite-time stability
analysis of neural networks with time-varying delays**

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Abstract: In this paper, we investigate the problem of the finite-time stability analysis of time-delayed neural networks. By constructing new Lyapunov-Krasovskii (LKF) functional with new integral terms sufficient condition derived for guarantee the finite-time stability with time-varying delays. By using the Jensen's inequality, reciprocally convex combination method Linear matrix inequalities (LMIs) criterion is obtained. Numerical examples are addressed to show the effectiveness of the proposed results.

Key Words: Finite-time stability, Linear matrix inequalities, Lyapunov-Krasovskii method, Neural networks, Time-varying delays.

AMS Subject Classification: 34A34, 34A37, 92B20, 34K20

1 Introduction

Over the past few decades, neural networks have been deeply discussed and have been found a wide range of applications in different fields, signal processing, image processing, optimization solver and other practical areas. Therefore, the stability is a fundamental issue for neural networks, which has been extensively studied; see, for instance, [1, 2], and the references therein.

The concept of finite-time stability, which was first introduced to the control field in the 1960s [3], is that, given a bound on the initial condition, the system's state does not exceed a certain threshold during a specified time interval. Finite-time stability shows the behavior of the system over a fixed finite-time interval. In fact, in

many practical applications, such as chemical processes and power systems, the main concern is the transient response of system states over a finite-time horizon. Note that many meaningful results on finite-time stability and finite-time boundedness have been developed in [4, 5] the past decades.

However, the best of our knowledge, new results on finite-time stability analysis of distributed time-delay neural networks has not yet been investigated in the literature. The major contributions of this paper are as follows: by constructing the Lyapunov function approach and the LMI technique, by using the reciprocally convex approach in [6] and the Jensen's inequality in [7], a new approach is developed to bound the derivative of the LK functional. Moreover, the we found less conservative results for compared to exists results. Finally, Numerical examples are presented to illustrate the efficiency of the proposed method and conclusion.

Notations: We let $\mathbb{R}^m$ denotes Euclidean space, m is real matrix: $\text{diag}\{W, B\}$ represents the block diagonal matrix of W and B ; I is the identity matrix with appropriate dimensions. $N > 0$ denote a symmetric positive definite matrix N , $\lambda_{\max}(N)$ and $\lambda_{\min}(N)$ denote the maximum and minimum eigenvalues of matrix N , respectively. $*$ represents the elements below the main diagonal of a symmetric matrix.

2 Problem Statement and Preliminaries

$$\left. \begin{aligned} \dot{x}(t) &= -Ax(t) + W_0f(x(t)) + W_1f(x(t - \tau(t))) \\ &\quad + C \int_{t-\bar{\rho}}^t f(x(s))ds, \\ x(t) &= \phi(t), \quad t \in [-H, 0] \end{aligned} \right\} \quad (1)$$

where $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T \in \mathbb{R}^n$ is the state vector associated with n neurons, $A = \text{diag}\{a_1, a_2, \dots, a_n\}$ is known diagonal matrices with positive entries. W_0 , W_1 and C are connection weight matrices; $f(x(t)) = [f_1(x_1(t)), \dots, f_n(x_n(t))]^T \in \mathbb{R}^n$ is the

neuron activation function and $\phi(t)$ is the given initial condition, $-H = \max\{\bar{\tau}, \bar{\rho}\}$; The time-varying delays $\tau(t)$ and $\rho(t)$ satisfy

$$0 \leq \tau(t) \leq \bar{\tau}, \quad \dot{\tau}(t) \leq \mu_1, \quad 0 \leq \rho(t) \leq \bar{\rho},$$

where $\bar{\tau}$, $\bar{\rho}$ and μ_1 are constants.

Assumption. 1. The neuron activation function $f_l(x(t))$ $l \in \{1, 2, \dots, n\}$ is bounded and satisfying the following inequality:

$$k_l^- \leq \frac{f_l(y_1) - f_l(y_2)}{y_1 - y_2} \leq k_l^+,$$

$$f_l(0) = 0, \quad y_1, y_2 \in \mathbb{R}, \quad y_1 \neq y_2, \quad l = 1, 2, \dots, n,$$

for all $y_1, y_2 \in \mathbb{R}$, with k_l^-, k_l^+ being known real constants with $l = 1, 2, \dots, n$.

Definition 1. [5] Given scalars $c_2 > c_1 \geq 0$, and the positive definite matrix R , system (1) is said to be finite-time stable with respect to (c_1, c_2, T, R) if

$$\sup_{-H \leq s \leq 0} \{x^T(s)Rx(s)\} \leq c_1 \Rightarrow x^T(t)Rx(t) < c_2, \forall t \in [0, T].$$

3 Main results

In this section, we will derive the finite-time stability of neural networks with distributed time-varying delays.

From Assumption 1

$$K_1 = \text{diag}\{\kappa_1^-, \kappa_2^-, \dots, \kappa_n^-\}, \quad K_2 = \text{diag}\{\kappa_1^+, \kappa_2^+, \dots, \kappa_n^+\},$$

$$K_3 = \text{diag}\{\kappa_1^-, \kappa_1^+, \dots, \kappa_n^-, \kappa_n^+\}, \quad K_4 = \frac{1}{2} \text{diag}\{\kappa_1^- + \kappa_1^+, \dots, \kappa_n^- + \kappa_n^+\}.$$

$$\xi(t) = \text{col} \left\{ x(t), x(t - \tau(t)), x(t - \bar{\tau}), \dot{x}(t), \int_{t-\tau(t)}^t x(s)ds, f(x(t)), \right.$$

$$\left. f(x(t - \tau(t))), \int_{t-\rho(t)}^t f(x(s)) \right\}, \quad e_k = [0_{(k-1)n \times n}, I_n, 0_{(k-1)n \times n}], \quad k = (1, 2, \dots, 8).$$

Theorem 2. Given scalars $\bar{\tau}$, $\bar{\rho}$, μ_1 , α and a time-constant $T > 0$ the time delayed neural networks (1) is said to be finite time stable with respect to (c_1, c_2, T) , given constant $\gamma > 0$, the diagonal matrices $W_i > 0, (i=1,2)$, $D > 0$ and symmetric matrices

$P > 0$ $S_i > 0$ ($i=1,2,3,4$), $M_1 > 0$ with appropriate dimensions such that the following matrix inequalities holds:

$$\begin{bmatrix} S_3 & M_1 \\ * & S_3 \end{bmatrix} \geq 0, \quad (2)$$

$$[\Sigma_{1ij}]_{(8 \times 8)} < 0, \quad (3)$$

$$c_1 \Lambda e^{\alpha T} < \lambda_1 c_2, \quad (4)$$

where

$$\Sigma_1 = \Phi_1 + \dots + \Phi_6, \Phi_1 = 2e_1 P e_4^T - 2\gamma e_1 P e_1^T + 4\gamma d_1 D^T e_6^T + 2e_4 D^T e_6^T,$$

$$\Phi_2 = e^{2\gamma\bar{\tau}} e_1 S_1 e_1^T - (1 - \mu_1) e_2 S_1 e_2^T + e^{2\gamma\bar{\tau}} e_6 S_2 e_6^T - (1 - \mu_1) e_7 S_2 e_7^T,$$

$$\Phi_3 = e_4 S_3 e_4^T + \begin{bmatrix} (e_1 - e_2)^T \\ (e_2 - e_3)^T \end{bmatrix}^T \begin{bmatrix} S_3 & M_1 \\ * & S_3 \end{bmatrix} \begin{bmatrix} (e_1 - e_2)^T \\ (e_2 - e_3)^T \end{bmatrix},$$

$$\begin{aligned} \Phi_4 = & \bar{\rho} e_6 S_4 e_6^T - \frac{e^{\gamma(t-\bar{\rho})}}{\bar{\rho}} e_8 S_4 e_8^T, \Phi_5 = 2e_4 Y e_4^T + 2e_1 A^T Y e_4^T \\ & - 2e_4 Y W_0 e_6^T - 2e_4 Y W_1 e_7^T - 2e_4 Y C e_8^T + 2e_1 A Y A^T e_1^T \\ & - 2e_1 A Y W_0^T e_6^T - 2e_1 A Y W_1^T e_7^T - 2e_1 A Y C^T e_8^T + 2e_6 W_0 Y W_0^T e_6^T \\ & + 2e_6 W_0 Y W_1^T e_7^T + 2e_6 W_0 Y C^T e_8^T + 2e_7 W_1 Y W_1^T e_7^T \\ & + 2e_6 W_1 Y C^T e_8^T + 2e_8 C Y C^T e_8^T, \end{aligned}$$

$$\begin{aligned} \Phi_6 = & -e_1 W_1 K_3 e_1^T + 2e_1 W_1 K_4 e_6^T - e_6 W_1 e_1^T - e_2 W_2 K_3 e_2^T \\ & + 2e_2 W_2 K_4 e_7^T - e_7 W_2 e_2^T. \end{aligned}$$

$$\begin{aligned} \lambda_1 = \lambda_{\min}(\tilde{P}) = \lambda_{\min}(R^{-\frac{1}{2}} P R^{-\frac{1}{2}}), \lambda_2 = \lambda_{\max}(\tilde{P}) = \lambda_{\max}(R^{-\frac{1}{2}} P R^{-\frac{1}{2}}), \\ \lambda_3 = \lambda_{\max}(\tilde{D}) = \lambda_{\max}(R^{-\frac{1}{2}} D R^{-\frac{1}{2}}), \lambda_i = \lambda_{\max}(\tilde{S}_i) = \lambda_{\max}(R^{-\frac{1}{2}} S_i R^{-\frac{1}{2}}), \\ i = (4, 5, 6, 7), j = (1, 2, 3, 4). \end{aligned}$$

$$\begin{aligned} \Lambda = & \lambda_2 e^{2\gamma t} + 2\lambda_3 e^{2\gamma t} |K_1, K_2| + 2\lambda_4 \bar{\tau} e^{2\gamma t} e^{2\gamma t} + \lambda_5 \bar{\tau} e^{2\gamma t} e^{2\gamma t} (|K_1, K_2|^2) + \\ & \lambda_6 \frac{\bar{\tau}^3}{2} e^{2\gamma t} + \lambda_7 \frac{\bar{\tau}^2}{2} e^{2\gamma t} |K_1, K_2|^2. \end{aligned}$$

Proof. Construct the LK functional candidate as follows:

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) + V_4(x_t) + V_5(x_t), \quad (5)$$

where

$$V_1(x_t) = e^{2\gamma t} x^T(t) P x(t) + 2 \sum_{i=1}^n d_i e^{2\gamma t} \int_0^{x_i(t)} f_i(s) ds,$$

$$\begin{aligned}
V_2(x_t) &= e^{2\gamma\bar{\tau}} \left[\int_{t-\tau(t)}^t e^{2\gamma s} x^T(s) S_1 x(s) ds + \int_{t-\tau(t)}^t e^{2\gamma s} f^T(s) S_2 f(s) ds \right], \\
V_3(x_t) &= \bar{\tau} \int_{-\bar{\tau}}^0 \int_{t+\theta}^t e^{2\gamma s} \dot{x}^T(s) S_3 \dot{x}(s) ds d\theta, \\
V_4(x_t) &= \int_{-\bar{\rho}}^0 \int_{t+\theta}^t e^{2\gamma s} f^T(x(s)) S_4 f(x(s)) ds d\theta.
\end{aligned}$$

Along the trajectories (1) we taking derivative on equation (5) and applying Jensen's inequality [7] and Reciprocal convex combination [6] Lemmas it yields:

$$\begin{aligned}
\dot{V}_1(x_t) &\leq 2e^{2\gamma t} x^T(t) P \dot{x}(t) + 2\gamma e^{2\gamma t} x^T(t) P x(t) + 4\gamma e^{2\gamma t} f^T(x(t)) D x(t) \\
&\quad + 2e^{2\gamma t} f^T(x(t)) D \dot{x}(t), \\
&= e^{2\gamma t} \xi^T(t) \Phi_1 \xi(t),
\end{aligned} \tag{6}$$

$$\begin{aligned}
\dot{V}_2(x_t) &\leq e^{2\gamma\bar{\tau}} e^{2\gamma t} x^T(t) S_1 x(t) - (1 - \mu_1) x^T(t - \tau(t)) S_1 x(t - \tau(t)) \\
&\quad + e^{2\gamma\bar{\tau}} e^{2\gamma t} f^T(x(t)) S_2 f(x(t)) \\
&\quad - (1 - \mu_1) e^{2\gamma t} f^T(x(t - \tau(t))) S_2 f(x(t - \tau(t))), \\
&\leq e^{2\gamma t} \xi^T(t) \Phi_2 \xi(t),
\end{aligned} \tag{7}$$

$$\begin{aligned}
\dot{V}_3(x_t) &= \bar{\tau}^2 e^{2\gamma t} + \dot{x}^T(t) S_3 \dot{x}(t) - \bar{\tau} \int_{t-\bar{\tau}}^t e^{2\gamma s} \dot{x}^T(s) S_4 \dot{x}(s) ds \\
&= e^{2\gamma t} \xi^T(t) \Phi_3 \xi(t),
\end{aligned} \tag{8}$$

$$\begin{aligned}
\dot{V}_4(x_t) &= e^{2\gamma t} \bar{\rho} f^T(x(t)) S_7 f(x(t)) - \int_{t-\rho(t)}^t e^{2\gamma s} f^T(x(s)) S_7 f(x(s)) ds, \\
&\leq e^{2\gamma t} \xi^T(t) \Phi_4 \xi(t).
\end{aligned} \tag{9}$$

For any matrices $Y > 0$ which are equal to zero and it can be defined with compatible dimensions,

$$\begin{aligned}
0 &= 2e^{2\gamma t} Y \left[\dot{x}(t) + A x(t) - W_0 f(x(t)) - W_1 f(x(t - \tau(t))) \right. \\
&\quad \left. - C \int_{t+\rho(t)}^t f(x(s)) ds \right], \\
&= \xi^T(t) \Phi_5 \xi(t).
\end{aligned} \tag{10}$$

Based on Assumption 1, the following matrix inequalities hold for any positive diagonal matrices $W_i (i = 1, 2)$

$$0 \leq e^{2\gamma t} \left[-x^T(t) W_1 K_3 x(t) + 2x^T(t) W_1 K_4 f(x(t)) \right]$$

$$\begin{aligned}
& -f^T(x(t))W_1f(x(t))\Big], \\
0 & \leq e^{2\gamma t}\Big[-x^T(t-\tau(t))W_2K_3x(t-\tau(t))+2x^T(t-\tau(t)) \\
& \quad \times W_2K_4f(x(t-\tau(t)))-f^T(x(t-\tau(t)))W_2f(x(t-\tau(t)))\Big], \\
& = \xi^T(t)\Phi_6\xi(t). \tag{11}
\end{aligned}$$

Thus adding the terms of (6)-(11) we obtain following inequality,

$$\dot{V}(x_t) = e^{2\gamma t}\xi^T(t)\Sigma_1\xi(t). \tag{12}$$

Hence, $\dot{V}(x_t) < 0$ can be held by $\Sigma_1 < 0$. On the other hand, from (12) and needed constant $\alpha > 0$, it yields that

$$\dot{V}(x_t) < \alpha V(x_t). \tag{13}$$

From above inequality differentiating on both sides, it yields

$$e^{-\alpha t}V(x_t) < V(x_0). \tag{14}$$

Note that $0 \leq t \leq T$, we can apply the initial condition taking from [5] obtain the following inequality

$$V(x_0) \leq c_1\Lambda e^{\alpha T}. \tag{15}$$

On other hand, we can get

$$V(x(t)) \geq x^T(t)Px(t) \geq \lambda_{\min}(\tilde{P})x^T(t)Rx(t) = \lambda_1x^T(t)Rx(t). \tag{16}$$

From (15) and (16) we can obtain

$$x^T(t)Rx(t) < \frac{c_1\Lambda e^{\alpha T}}{\lambda_1}. \tag{17}$$

By the condition (4), we can get

$$x^T(t)Rx(t) < c_2. \tag{18}$$

By Definition 2.1, we conclude that system (1) is finite-time stable with respect to (c_1, c_2, T) . $\square$

Remark 3. From the neural networks (1) we reduced to without distributed delay (i.e. $C = 0$). Then the system is said to be asymptotically stable.

4 Numerical examples

This section we presented the numerical examples.

Example 4.1. Consider neural networks (1) with time varying delays and the following matrices $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, $W_0 = \begin{bmatrix} 1.3 & 0.4 \\ 0.9 & 0.2 \end{bmatrix}$, $W_1 = \begin{bmatrix} 0.6 & 0.2 \\ 0.3 & 0.5 \end{bmatrix}$, $C = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & -0.3 \end{bmatrix}$, $K_1 = I_2$, $K_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\bar{\tau} = 1.6$, $\bar{\rho} = 1.1$, $\mu_1 = 0.7$, $\mu_2 = 0.3$, $T = 5$, $c_1 = 3$, $\alpha = 0.001$, $\gamma = 0.03$. By using the MATLAB LMI toolbox, solve the LMIs in Theorem 3, we get the unknown constant $c_2 = 5.6$. Thus, the system is finite-time stable with respect to (c_1, c_2, T) .

Example 4.2. Consider the delayed neural networks (1) with ($C=0$) Compared with the previous results proposed in [1], we less conservative than the one in [1](see Table 1). Also, we have shown state responses of the system (1) with ($C=0$) in Figure.1.

Figure 1: State trajectories of neural networks (1) with ($C=0$)

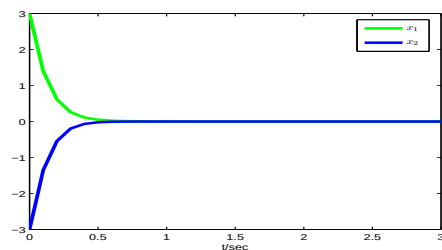


Table 1: Maximum allowable bound $\bar{\tau}$ for different values μ_1 in Example 4.2.

Methods	$\mu_1 = 0.8$	$\mu_1 = 0.9$	Unknown μ_1
[1]	2.9541	1.9654	1.7839
Remark 5	8.1047	6.0436	5.7825

5 Conclusion

This paper has been investigated the new criterion delay dependent finite-time stability of neural networks with distributed time-varying delays. Constructing LK functionals and deals with Jensen's inequality, Reciprocally convex combination methods. A LMI approach has been utilized to establish the sufficient conditions for the finite time stability. Also, we compared with existing results and has less conservative results. These conditions can be readily checked by utilizing the Matlab LMI toolbox. Finally, numerical examples has been provided to demonstrate the usefulness of the proposed methods.

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