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Survey on Total Variation based Image Regularization Algorithms for Image Denoising

V Kamalaveni[1*], K A Narayanankutty[2], S. Veni[3], K. P. Soman[4]
[1]Department of Computer Science and Engineering, Amrita School of Engineering, Coimbatore, Amrita Vishwa Vidyapeetham, India

[1,2,3]Department of Electronics and Communication Engineering, Amrita School of Engineering, Coimbatore, Amrita Vishwa Vidyapeetham, , India, ka_narayanankutty@cb.amrita.edu, kp_soman@amrita.edu, s_veni@cb.amrita.edu

*Corresponding author: Tel.+91-956-661-8469 Email Address: vvkamalaveni@gmail.com

Abstract---This paper is devoted to literature review of total variation based models to image denoising and restoration. In these image denoising techniques the total variation of an image is minimized. Due to the increased need for good quality image in several applications, an extensive research work has been done in the area image regularization with special focus on image denoising. Total variation denoising is a mathematical approach developed to remove noise in an image and simultaneously preserve sharp edges in an image. Unlike linear filters, total variation denoising is formulated as an optimization problem. This method of denoising produces the denoised image by minimizing an energy functional or cost function. Therefore any numerical algorithm that can solve the optimization problem can be used to implement total variation based image denoising. Total variation minimization concept is used not only for denoising, but also for several image restoration methods such as in-painting, interpolation, deblurring and compressed sensing.

Keywords— **Total Variation; Energy functional; optimixation; gradient;edge function**

I. Introduction

In image processing total variation based image regularization finds its application in noise removal. The basic principle behind this technique is that images with lots of erroneous details have maximum total variation. Total variation is computed by adding up the gradient of the image at every pixel position. Based on this concept, minimizing the total variation of the image subject to the condition that the processed image closely matches with the original image can eliminate the undesirable details concurrently preserving essential information such as edges and corners. The total variation based noise removal has benefits over linear smoothing filtering which can eliminate noise but smoothes edges and corners whereas total variation based image denoising is very effective in smoothing noise in flat inner region simultaneously preserving edges.

For a two dimensional image u, the image TV can be computed using equation 1[2][16][18] .

$$TV(u) = \int_{\Omega} |\nabla u| \, d\Omega \tag{1}$$

Given a` noisy image u_0, the method of denoising using total variation is to find an image u having minimum total variation than u_0 at the same time closely matches with u_0. The closeness is measured by the fidelity term as the sum of the squared errors. So the image denoising problem based on total variation is formulated as minimizing the functional given in Eq. 3 [13][19][20].

$$fidelityterm = \int_{\Omega} |(u - u_0)|^2 \, d\Omega \tag{2}$$

$$\min_u \int_\Omega |\nabla u| d\Omega + \lambda \int_\Omega (u - u_0)^2 d\Omega \qquad (3)$$

The regularization parameter λ has a greater significance in image denoising. When λ=0 the amount of denoising is zero, the processed image becomes equal to original noisy image. When λ becomes infinity the total variation term influences and the effect is processed image has a lower value of total varition at the same time processed image does not look like an original input image. Therefore the value of regularization parameter λ is very important for achieving effective denoising[12][15][21]. The variational model can be solved using the numerical algorithms mentioned in the papers[13][17][22]. Though the total variation based regularization can able to minimize noise and also regularize the edges, it has the following undesirable properties.

Loss of contrast: When the total variation of an image is minimized, the contrast of an image that is the difference between the highest pixel value and the lowest pixel value is reduced. ROF model cannot preserve corners and contrast of the image and ultimately the corners will be smeared. Loss of Geometry: The total variation of an image can also be reduced by minimizing the length of level sets. This process will result in distortion of the geometry of the level sets. Loss of Texture: Though total variation regularization is very effective for denoising, this method can not be able to preserve texture. This is because of the above mentioned properties contrast and geometry loss both will affect small scale features like texture very severely. Staircase Effect: This means that the denoised image may have blocky regions that is piecewise constant regions.

II TOTAL VARIATION-TIKHONOV MODEL

Tikhonov regularization method is the basic regularization method, the energy functional corresponding to this model described by eqn.4 contains L2-norm of the image gradient whereas in ROF model L1-norm is used.

The gradient descent method can be used to solve the Euler's Lagrangian equation of the functional used in variational methods. The partial differential equation determined by the gradient descent method can be solved using Euler's

$$\min_u \{E(u)\} = \min_u \int_\Omega |\nabla u(x,y)|^2 d\Omega + \lambda \int_\Omega (u - u_0)^2 d\Omega \qquad (4)$$

forward finite difference method. The resulting PDE is given below.

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial u} = (u_{xx} + u_{yy}) - \lambda(u - u_0) \qquad (5)$$

Tikhonov regularization method enables strong smoothing and not able to preserve important geometric features of an image such as edges, corners and texture.

III TOTAL VARIATION-ROF MODEL

Rudin, Osher and Fatemi in 1992 used total variation concept for image denoising. The ROF model[1] is described by a functional given in Eq. 6.

Here u_0 is the given input noisy image and u is the processed image produced by the denoising

$$\min_u \{E(u)\} = \min_u \int_\Omega |\nabla u(x,y)| d\Omega + \lambda \int_\Omega (u - u_0)^2 d\Omega \qquad (6)$$

algorithm. Here λ is the regularization parameter. The following partial differential equation is obtained when the above functional is solved using gradient descent method[22].

The regularization term can be either isotropic total variation or anisotropic total variation. This

$$\frac{\partial u}{\partial t} = -\frac{\partial E}{\partial u} = \frac{u_{xx}u_y^2 - 2u_x u_y u_{xy} + u_{yy}u_x^2}{(u_x^2 + u_y^2)^{3/2}} - 2\lambda(u - u_0) \qquad (7)$$

model preserves edges properly while removing noise, but it can not be able to preserve texture, contrast and geometrical features. This model also causes staircasing effect.

IV TOTAL VARIATION- TV L1

ROF model is a convex optimization model which has a unique global minimizer whereas TV-L1 model is not-strictly convex optimization model

$$\min_u \{E(u)\} = \min_u \int_\Omega |\nabla u(x,y)| d\Omega + \lambda \int_\Omega |u - u_0| d\Omega \qquad (8)$$

and the global minimizer is not unique. Interesting features of TV-L1 model are contrast preservation and geometry preservation. The functional in eqn.8 describes the TV L1 model. The minimizer for the above functional can be obtained by solving the PDE given below.

$$\frac{\partial u}{\partial t} = \frac{u_{xx}u_y^2 - 2u_x u_y u_{xy} + u_{yy}u_x^2}{\left(u_x^2 + u_y^2\right)^{3/2}} - \lambda \frac{(u-u_0)}{|u-u_0|} \quad (9)$$

The iterative numerical algorithm using euler's forward finite difference method is used to solve the partial differential equations 5, 7, 9 and denoised images are produced as per the steps described in paper authored by V.Kamalaveni et al.[2]. The algorithm removes the noise present in the image and concurrently preserves edges using certain number of iterations.

V ADAPTIVE TOTAL VARIATION- DAVID STRONG MODEL

David Strong[3] introduced the adaptive total variation model as the following functional. Here $g(|\nabla u|)$ is the edge stopping function whose value is insignificant or zero near edges[23]. This edge function has value one in the inner region of the image.

$$\min_u \{E(u)\} = \min_u \int_\Omega g(|\nabla u|)|\nabla u(x,y)| d\Omega + \frac{\lambda}{2}\int_\Omega (u-u_0)^2 d\Omega \quad (10)$$

VI DIFFERENCE OF CURVATURE BASED ADAPTIVE TV MODEL

Qiang Chen et al. [4] proposed an adaptive TV algorithm making use of new edge stopping function based on difference of curvature. In this algorithm regularization term as well as fidelity term changes adaptively depending on whether the pixel belongs to an edge or flat region. At image edges the regularization term is set to total variation of the image and for the flat region the regularization term is set to square of the gradient. At the object boundaries the regularization parameter λ is set to higher value and for the flat region the regularization parameter λ is set to lower value. The algorithm is found to be very efficient in eliminating rician noise present in the MRI images. The difference of curvature(D) is defined as follows.

$$D = \|u_{\eta\eta}\| - \|u_{\xi\xi}\|$$

Here $u_{\eta\eta}$ is the directional second derivative along gradient direction and $u_{\xi\xi}$ is the directional second derivative along the direction perpendicular to gradient direction that is along edge direction.

VII TOTAL VARIATION USING HIGHER ORDER PDE

Fourth order PDEs are good in eliminating staircasing effect but not very efficient in preserving edges. Lysaker et al. in 2003 proposed a total variation denoising model using fourth order partial differential equation as follows[5].

Here $|\nabla^2 u|$ is the Laplacian operator. The problem with higher order PDE is edges are not preserved

$$\min_u \{E(u)\} = \min_u \int_\Omega (|\nabla^2 u|) d\Omega + \frac{\lambda}{2}\int_\Omega (u-u_0)^2 d\Omega \quad (11)$$

and are blurred.

Weifeng Zhou and Qingguo Li proposed a fourth order PDE based TV regularization algorithm for removing poisson noise[6]. They used alternating minimization algorithm for solving the proposed functional given in Eq. 12[14].

$$\min_u \{E(u)\} = \min_u \int_\Omega (|\nabla^2 u|) d\Omega + \frac{\lambda}{2}\int_\Omega (u - u_o \log u) d\Omega \quad (12)$$

VIII HYBRID MODEL

Li et al. in 2007 proposed an adaptive hybrid regularizer using both TV model and LLT model proposed by Lysaker et al.[7]

$$\min_u \{E(u)\} = \min_u \int_\Omega ((1-g(x))|\nabla u| + g(x)|\nabla^2 u|) d\Omega + \frac{\lambda}{2}\int_\Omega (u-u_0)^2 d\Omega$$

Here g(x) is an edge indicator whose value is zero near edges and one in the inner region. So at the edges the regularization term becomes gradient of an image so model behaves like a ROF model thereby able to preserve edges properly. In the flat inner region the regularization term becomes square of the gradient so the model behaves like a tikhonov model. So smoothing is strong in the flat region.

IX LIMING TANG AND ZHUANG FANG MODEL

Liming Tang and Zhuang Fang proposed a forward diffusion also backward diffusion in total variation regularization algorithm[8]. This algorithm is very efficient in preserving boundaries and image contrast.

The algorithm is described by functional

$$\min_u\{E(u)\} = \alpha\int_\Omega |\nabla u| dx + \beta\int \varphi(|\nabla u|)dx + \frac{1}{2}\int_\Omega (u-u_0)^2 d\Omega \qquad (13)$$

A proper balance between backward and forward diffusion can be achieved using the parameters α and β. The function φ must satisfy the following three conditions. (c1) φ(s) >=0 for any s ϵ (0, +∞); (c2) φ(s) is monotonically decreasing function in (0, +∞); (c3) φ(0) = 1 and Limit $_{s\to\infty}$ φ(s) = 0;

The descent energy flow related to the fuctional (13) is

$$\frac{\partial u}{\partial t} = \alpha div\left(\frac{\nabla u}{|\nabla u|}\right) + \beta div\left(\frac{\varphi'(|\nabla u|)}{|\nabla u|}\nabla u\right) + (u-u_0) \qquad (14)$$

The partial differential equation equivalent to the equation 14 is given in equation 15.

$$\frac{\partial u}{\partial t} = div\left(\frac{\alpha + \beta\varphi'(|\nabla u|)}{|\nabla u|}\nabla u\right) + (u-u_0) \qquad (15)$$

In equation 15 $\frac{\alpha+\beta\varphi'(|\nabla u|)}{|\nabla u|}$ is the diffusion coefficient and here the term α/|▽u| > 0 is a forward diffusion coefficient which decides the amount of smoothing for diffusion equation 15.

The term βφ$^{/}$(|▽u|)/|▽u| < 0 is a backward diffusion coefficient which enables the sharpness and contrast enhancement for the diffusion equation 15.

X FENLIN YANG MODEL

Fenlin Yang et al. proposed adaptive second order variational model[9]. The adaptivity is achieved using the following functional described by eqn.16.

$$\int_\Omega \alpha(x,y)|\nabla u|\min(\theta,|\nabla u|)dxdy \qquad (16)$$

Where θ is the small non-negative parameter and

$$\alpha(x,y) = \left(\frac{1}{\max(\theta,|\nabla G * u|)}\right) \qquad (17)$$

Here G ia a gaussian filter. α has higher value away from edges and has smaller value near the edges. |▽u| is used to detect the presence of edges. When |▽u| is greater than θ this implies the existence of an edge at that pixel position and the model adaptively becomes TV model. When the |▽u| is smaller than θ, then the pixel might be part of flat inner region and the regularization term adaptively becomes the L2 norm to smooth away the noise present in the inner region.

$$\min(\theta,|\nabla u|) = \frac{\theta + |\nabla u| - |\theta - |\nabla u||}{2} \qquad (18)$$

Finally second order unconstrained adaptive TV denoising model becomes

$$\min_u\{E(u)\} = \min_u \int_\Omega \alpha(x,y)\left(\theta|\nabla u| + |\nabla u|^2 - \left|\theta|\nabla u| - |\nabla u|^2\right|\right) + \frac{1}{2}(u-u_0)^2 dxdy$$

XI KUI LUI HYBRID REGULARIZER

Liu et al. proposed hybrid regularizer for adaptive anisotropic diffusion[7].In the proposed model H^{-1} norm is used as fidelity term. The fourth order PDE and TV norm are combined to form regularizing term. At image boundaries the total variation filter is used which can preserve edges and in the flat region fourth order filter is selected which enables strong image smoothing. The novel

$$\min_u\{E(u)\} = \min_u \int_\Omega ((1-g(x))|\nabla u| + g(x)|\nabla^2 u|)d\Omega + \frac{\lambda}{2}\int_\Omega \|u-u_0\|^2_{H^{-1}} d\Omega$$

adaptive TV is defined by the above equation. Here H^{-1} norm is more efficient in preserving textures and oscillatory patterns. The authors used split-Bregman method to solve the equation 20.

XII QiIANGQIANG YUAN MODEL

The authors Qiangqiang Yuan et al. proposed a TV model for hyperspectral image denoising[10]. The algorithm uses the split Bregman iteration technique for solving the TV model. The straight forward method of using TV model to hyperspectral images is to use the TV model to every band in an image. TV is determined by finding TV for each band and then add the computed TV of each band. The band-by-band hyperspectral TV is described as follows.

$$HTV(u) = \sum_{j=1}^{B} TV(u_j) \quad (19)$$

Where u_j stands for the jth band of hyperspectral image. u is the hyperspectral image and it consist of different bands u = { u_1,u_2, u_3 ... u_B}.

The hyperspectral TV is further described as follows.

$$HTV(u) = \sum_{i=1}^{MN} \sqrt{\sum_{j=1}^{B} (\nabla_{i,j} u)^2} \quad (20)$$

$$(\nabla_{i,j} u)^2 = (\nabla_{i,j}^{h} u)^2 + (\nabla_{i,j}^{\upsilon} u)^2 \quad (21)$$

Where MN is the number of pixels in one band and B is the number of bands. The hyperspectral denoising model can now be formulated as constrained optimization problem as follows. $\nabla_{i,j}^{h} u$ is the gradient along horizontal direction at i^{th} pixel in the j^{th} band and $\nabla_{i,j}^{v} u$ is the gradient along vertical direction at i^{th} pixel in the j^{th} band.

$$\hat{u} = \arg\min \left\{ \sum_{j=1}^{B} \| u_j - f_j \|_2^2 + \lambda \sum_{i=1}^{MN} \sqrt{\sum_{j=1}^{B} (\nabla_{i,j} u)^2} \right\} \quad (22)$$

Euler lagrangian equation of the above equation 22 is as follows.

$$(u_j - f_j) - \lambda \nabla \cdot \frac{\nabla u_j}{\sqrt{\sum_{j=1}^{B} (\nabla u_j)^2}} = 0$$

XIII YAO ZHAO MODEL - SAR IMAGE DESPECKLING

Yao Zhao et al. proposed an adaptive total variation regularization model for removing speckle noise from radar images which estimates noise level by means of wavelets[11]. Chambolle's method is used to solve the total variation regularization model. In the adaptive model the regularization parameter λ is dependent on noise level and current estimation of the despeckled image. This adaptive TV model becomes minimization problem given by equation 26.

$$\min_{f} TV(f) \quad \textit{such that} \quad \| f - u \| \leq \varepsilon \quad (26)$$

Where $\in = \sqrt{N}\,\sigma$ is the noise level and N is the number of pixels of the SAR image. σ is the noise variance of the image. The λ at each step is updated as follows.

$$\lambda^{k+1} = \frac{\lambda^k \sqrt{N}\sigma}{\| f^k - u \|}$$

XIV TV-MEANS IMAGE DENOISING ALGORITHM

In this algorithm the image is divided into patches and each patch is denoised using TV regularization.

The following are the steps of TV-means algorithm.

Step1: Total-variation based denoising is applied to every image patch at various scales

Step2: For every pixel determine enough number of patches similar to current patch.

Step3: Find the average of these patches to obtain the denoised patch.

Step4: Combine the denoised patches to compute the denoised image.

This denoising algorithm makes use of two powerful denoising algorithms. One is total variation denoising. Another one is Non-Local means denoising. The features of both the algorithms are used in TV-means denoising algorithm. This algorithm produces good quality image than the total variation denoising and non-local means denoising.

The denoised images produced by basic TV algorithms ROF model, TV-L1 model and Tikhonov model are compared in Fig.1-3[2].

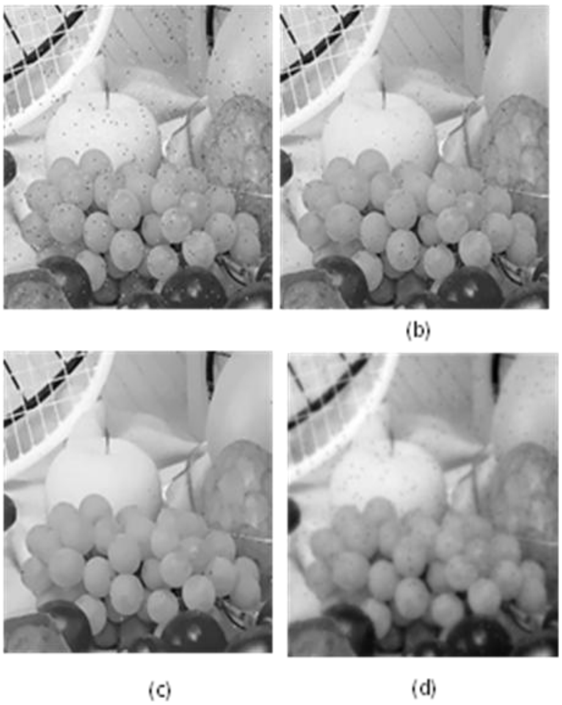

Fig.1. (a) noisy image (b) denoised image ROF model λ=0.2 (b) denoised image TV-L1 model λ=0.2 (d) denoised image Tikhonov model λ=0.2

Fig 3. (a) noisy image (b)denoised image Tikhonov model λ=0.2 (b) denoised image TV-L1 model λ=0.2 (d) denoised image ROF model λ=0.

Table-I shows the quality metrics computed for the denoised images produced by the basic TV based algorithms such as ROF model, TV-L1 model and Tikhonov model. From the visual quality of the denoised images and also from the values of quality mterics we can conclude that ROF model out performs other two algoritms[2].

XV CONCLUSION

We have summarized most recent total variation based regularization algorithms using partial differential equation for different noise models such as gaussian noise, salt and pepper noise, speckle noise and poission noise. We hope that the development of new total variation based regularization algorithm that spatially

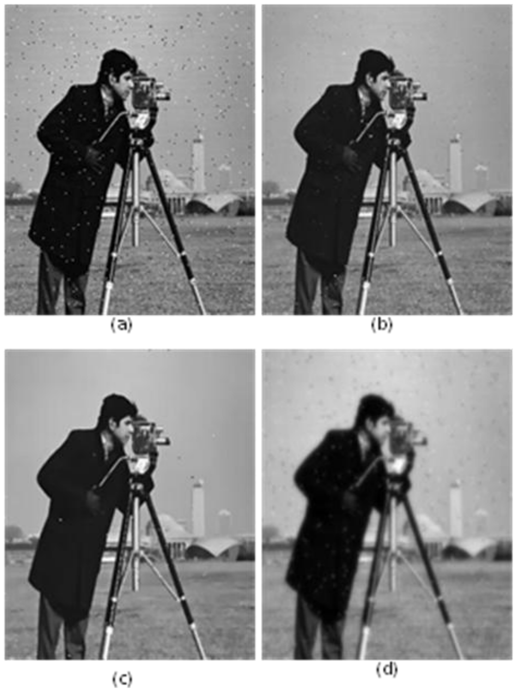

Fig 2. (a) noisy image (b) denoised image ROF model λ=0.2 (b) denoised image TV-L1 model λ=0.2 (d) denoised image Tikhonov model λ=0.2

TABLE I
PSNR AND SSIM FOR ROF MODEL, TV-L1, TIKHONOV λ=0.2

Image	ROF	TV-L1	TIKho	ROF	TV-L1	Tikho
	SSIM	SSIM	SSIM	PSNR	PSNR	PSNR
lena	0.91	0.75	0.70	30.26	24.57	21.55
fruits	0.89	0.69	0.66	28.68	24.69	21.02
Camera	0.83	0.69	0.65	26.85	23.61	20.02
coins	0.85	0.72	0.63	27.01	22.66	18.15
ship	0.88	0.72	0.70	29.38	24.36	20.40
Lift	0.83	0.65	0.70	28.74	24.45	25.84

adapt regularizing term as well as regularization parameter will overcome shortcomings in the current TV reconstruction algorithms. Total variation based regularization algorithms can be used in different fields ranging from astronomical to SAR based systems. TV based regularization algorithms are also used in medical applications such as MRI, ultrasound, PET and other image enhancement.

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