THE TRIPLE ENTIRE OF INTUITIONISTIC IDEAL OF FUZZY REAL NUMBERS OVER $p-$METRIC SPACES DEFINED BY MUSIELAK ORLICZ FUNCTIONS

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Abstract. In this article we introduce the intuitionistic sequence spaces $\Gamma_{IF(\mu,\eta)}^{3F}$ and $\Lambda_{IF(\mu,\eta)}^{3F}$, and study some basic topological and algebraic properties of these spaces. Also we investigate the relations related to these spaces and some of their properties like solidity, symmetricity, convergence free etc., and also investigate some inclusion relations related to these spaces.

1. Introduction

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ (or $\mathbb{C}$), where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [12,13], Esi et al. [2-5], Datta et al. [6], Subramanian et al. [14,15], Debnath et al. [7] and many others.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$\sup_{m,n,k} |x_{mnk}|^\frac{1}{m+n+k} < \infty.$$ 

The space of all triple analytic sequences are usually denoted by $\Lambda^3$. A triple sequence $x = (x_{mnk})$ is called triple entire sequence and it is denoted by $\Gamma^3$, if

$$|x_{mnk}|^\frac{1}{m+n+k} \to 0 \text{ as } m, n, k \to \infty.$$ 

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The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [8] as follows

\[ Z(\Delta) = \{ x = (x_k) \in w : (\Delta x_k) \in Z \} \]

for \( Z = c, c_0 \) and \( \ell_\infty \), where \( \Delta x_k = x_k - x_{k+1} \) for all \( k \in \mathbb{N} \).

The difference triple sequence space was introduced by Debnath et al. (see [8]) and is defined as

\[ \Delta^3 x_{mnk} = x_{mnk} - x_{m,n,k+1} - x_{m+1,n,k} + x_{m+1,n,k+1} - x_{m,n,k+1} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1} \]

and \( \Delta^0 x_{mnk} = \langle x_{mnk} \rangle \).

1.1. Definition. An Orlicz function (see [1, 9]) is a function \( M : [0, \infty) \rightarrow [0, \infty) \) which is continuous, non-decreasing and convex with \( M(0) = 0, M(x) > 0 \), for \( x > 0 \) and \( M(x) \rightarrow \infty \) as \( x \rightarrow \infty \). If convexity of Orlicz function \( M \) is replaced by \( M(x + y) \leq M(x) + M(y) \), then this function is called modulus function.

Let \( M \) and \( \Phi \) are mutually complementary Orlicz functions. Then, we have:

(i) For all \( u, y \geq 0 \),

\[ uy \leq M(u) + \Phi(y), \quad (\text{Young's inequality}) \]

(ii) For all \( u \geq 0 \),

\[ u \eta(u) = M(u) + \Phi(\eta(u)) \]

(iii) For all \( u \geq 0 \), and \( 0 < \lambda < 1 \),

\[ M(\lambda u) \leq \lambda M(u) \]

Lindenstrauss and Tzafriri [10] used the idea of Orlicz functions to construct Orlicz sequence space

\[ \ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\} \]

The space \( \ell_M \) with the norm

\[ \| x \| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\} \]

becomes a Banach space which is called an Orlicz sequence space. For \( M(t) = t^p \) \((1 \leq p < \infty)\), the spaces \( \ell_M \) coincide with the classical sequence space \( \ell_p \).

A sequence \( f = (f_{mnk}) \) of Orlicz function is called a Musielak-Orlicz function. A sequence \( g = (g_{mnk}) \) defined by
is called the complementary function of a Musielak-Orlicz function \( f \). For a given Musielak Orlicz function \( f \), the Musielak-Orlicz sequence space \( \mathcal{M}_f \) is defined by \([\text{see } 11]\)

\[
\mathcal{M}_f (|x_{mnk}|^{1/m+n+k}) \to 0 \text{ as } m, n, k \to \infty,
\]

where \( \mathcal{M}_f \) is a convex modular defined by

\[
\mathcal{M}_f (x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk} \left( |x_{mnk}|^{1/m+n+k} \right).
\]

We consider \( \mathcal{M}_f \) equipped with the Luxemburg metric

\[
d(x, y) = \sup_{m, n, k} \left\{ \inf \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{mnk} \left( \frac{|x_{mnk}|^{1/m+n+k}}{\rho} \right) \right) \leq 1 \}.
\]

Let \( w^3 \) denote the set of all complex triple sequences \( x = (x_{mnk})_{m,n,k=1}^{\infty} \) and \( \mathcal{M} : [0, \infty) \to [0, \infty) \) be an Orlicz function. Given a triple sequence, \( x \in w^3 \). Define the sets

\[
\Gamma^3_M = \left\{ x \in w^3 : \left( M \left( \frac{|x_{mnk}|^{1/m+n+k}}{\rho} \right) \right) \to 0 \text{ as } m, n, k \to \infty \text{ for some } \rho > 0 \right\}
\]

and

\[
\Lambda^3_M = \left\{ x \in w^3 : \sup_{m,n,k \geq 1} \left( M \left( \frac{|x_{mnk}|^{1/m+n+k}}{\rho} \right) \right) < \infty \text{ for some } \rho > 0 \right\}.
\]

The space \( \Lambda^3_M \) is a metric space with the metric

\[
d(x, y) = \inf \left\{ \rho > 0 : \sup_{m,n,k \geq 1} \left( M \left( \frac{|x_{mnk} - y_{mnk}|^{1/m+n+k}}{\rho} \right) \right) \leq 1 \right\}
\]

The space \( \Gamma^3_M \) is a metric space with the metric

\[
\tilde{d}(x, y) = \inf \left\{ \rho > 0 : \sup_{m,n,k \geq 1} \left( M \left( \frac{|x_{mnk} - y_{mnk}|^{1/m+n+k}}{\rho} \right) \right) \leq 1 \right\}
\]

Let \( n \in \mathbb{N} \) and \( X \) be a real vector space of dimension \( w \), where \( n \leq m \). A real valued function \( d_p(x_1, \ldots, x_n) = \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p \) on \( X \) satisfying the following four conditions:

(i) \( \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = 0 \) if and only if \( d_1(x_1, 0), \ldots, d_n(x_n, 0) \) are linearly dependent,

(ii) \( \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p \) is invariant under permutation,

(iii) \( \|(\alpha d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p = |\alpha| \|(d_1(x_1, 0), \ldots, d_n(x_n, 0))\|_p, \alpha \in \mathbb{R} \)

(iv) \( d_p((x_1, y_1), (x_2, y_2) \ldots , (x_n, y_n)) = (dx_1 x_2 \ldots x_n)^p + dy_1 y_2 \ldots y_n)^{1/p} \) for \( 1 \leq p < \infty; \) (or)

(v) \( d((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) := \sup \{ d_X(x_1, x_2, \ldots x_n), d_Y(y_1, y_2, \ldots y_n) \} \),
when \( x_1, x_2, \cdots, x_n \in X, y_1, y_2, \cdots, y_n \in Y \), is called the \( p \) product metric of the Cartesian product of \( n \) metric.

## 2. Definition and Preliminaries

A triple sequence \( x = (x_{mnk}) \) is said to possess limit 0 (denoted by \( P-\lim x = 0 \), ) if \( |x_{mnk}|^{1/m+n+k} \to 0 \) as \( m, n, k \to \infty \). In this case, we say it to be \( P- \) convergent 0.

### 2.1. Definition
A binary operation \( * : [0, 1] \times [0, 1] \times [0, 1] \to [0, 1] \times [0, 1] \times [0, 1] \) is said to be continuous with co-metric if it satisfies the following conditions:
1. \( * \) is associative and commutative,
2. \( a \ast 1 = a \) for all \( a \in [0, 1] \),
3. \( a \ast c \leq b \ast d \) whenever \( a \leq b \) and \( c \leq d \) for each \( a, b, c, d \in [0, 1] \). For example \( d(a, b) = a \cdot b = a \cdot b = d(b, a) \).

### 2.2. Definition
A binary operation \( \delta : [0, 1] \times [0, 1] \times [0, 1] \to [0, 1] \times [0, 1] \times [0, 1] \) is said to be continuous with metric if it satisfies the following conditions:
1. \( \delta \) is associative and commutative,
2. \( a \delta 0 = a \) for all \( a \in [0, 1] \),
3. \( a \delta c \leq b \delta d \) whenever \( a \leq b \) and \( c \leq d \) for each \( a, b, c, d \in [0, 1] \).

### 2.3. Note
The five tuple \((X, \mu, \eta, *, \delta)\) is said to be an intuitionistic fuzzy metric space (for short, IFMS) if \( X \) is a vector space, \( * \) is a continuous metric, \( \delta \) is a continuous co-metric and \( \mu, \eta \) are fuzzy sets on \( X \times X \times (0, \infty) \times (0, \infty) \times (0, \infty) \).

### 2.4. Definition
A family \( I \subset 2^{Y \times X \times Y} \) of subsets of a non-empty set \( Y \) is said to be an ideal in \( Y \) if
1. \( \phi \in I \)
2. \( A, B \in I \) imply \( A \cup B \in I \)
3. \( A \in I, B \subset A \) imply \( B \in I \).

while an admissible ideal \( I \) of \( Y \) further satisfies \( \{x\} \in I \) for each \( x \in Y \). Given \( I \subset 2^{N \times N \times N} \) be a non trivial ideal in \( \mathbb{N} \times \mathbb{N} \times \mathbb{N} \) and \((X, \mu, \eta, *, \delta)\) be an IFMS. A sequence \((x_{mn})_{m,n,k\in N \times N \times N} \) in \( X \) is said to be \( I- \) convergent to 0 in \( X \) with respect to the intuitionistic fuzzy metric \((\mu, \eta)\) if for each \( \epsilon > 0 \) and \( t > 0 \) the set \( A(\epsilon) = \{m, n \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \mu(x_{mnk} - 0, t, \|d_1(x_1), \cdots, d_n(x_n)\|_\mu) \geq 1 - \epsilon \} \) belongs to \( I \).

or

\[ A(\epsilon) = \{m, n \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \eta(x_{mnk} - 0, t, \|d_1(x_1), \cdots, d_n(x_n)\|_\mu) \leq \epsilon \} \] belongs to \( I \).
2.5. Definition. A non-empty family of sets $F \subset 2^{X \times X \times X}$ is a filter on $X$ if and only if

1. $\phi \in F$
2. for each $A, B \in F$, we have imply $A \cap B \in F$
3. each $A \in F$ and each $A \subset B$, we have $B \in F$.

2.6. Definition. An ideal $I$ is called non-trivial ideal if $I \neq \phi$ and $X \notin I$. Clearly $I \subset 2^{X \times X \times X}$ is a non-trivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on $X$.

2.7. Definition. A non-trivial ideal $I \subset 2^{X \times X \times X}$ is called (i) admissible if and only if $\{x : x \in X\} \subset I$. (ii) maximal if there cannot exists any non-trivial ideal $J \neq I$ containing $I$ as a subset.

If we take $I = I_f = \{A \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} : A is a finite subset \}$. Then $I_f$ is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the usual convergence. If we take $I = I_f = \{A \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \delta(A) = 0\}$ where $\delta(A)$ denote the asymptotic density of the set $A$. Then $I_f$ is a non-trivial admissible ideal of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and the corresponding convergence coincides with the statistical convergence.

Let $D$ denote the set of all closed and bounded intervals $X = [x_1, x_2, x_3]$ on the real line $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$. For $X, Y, Z \in D$, we define $X \leq Y \leq Z$ if and only if $x_1 \leq y_1 \leq z_1$, $x_2 \leq y_2 \leq z_2$ and $x_3 \leq y_3 \leq z_3$, $d(X, Y) = \max \{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|\}$, where $X = [x_1, x_2, x_3]$ and $Y = [y_1, y_2, y_3]$.

Then it can be easily seen that $d$ defines a metric on $D$ and $(D, d)$ is a complete metric space. Also the relation $\leq$ is a partial order on $D$. A fuzzy number $X$ is a fuzzy subset of the real line $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ i.e. a mapping $X : R \rightarrow J (= [0, 1])$ associating each real number $t$ with its grade of membership $X(t)$.

2.8. Definition. A fuzzy number $X$ is said to be (i) convex if $X(t) \geq X(s) \land X(r) = \min \{X(s), X(r)\}$, where $s < t < r$. (ii) normal if there exists $t_0 \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ such that $X(t_0) = 1$. (iii) upper semi-continuous if for each $\epsilon > 0, X^{-1}([0, a + \epsilon])$ for all $a \in [0, 1]$ is open in the usual topology of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Let $R(J)$ denote the set of all fuzzy numbers which are upper semicontinuous and have compact support, i.e. if $X \in R(J) \times R(J) \times R(J)$ the for any $\alpha \in [0, 1], [X]^{\alpha}$ is compact, where $[X]^{\alpha} = \{t \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : X(t) \geq \alpha, if \alpha \in [0, 1]\}$, $[X]^{0}$ = closure of
\( \{ t \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : X(t) > \alpha, \text{if} \alpha = 0 \} \).

The set \( \mathbb{R} \) of real numbers can be embedded \( \mathbb{R}(J) \) if we define \( \bar{r} \in \mathbb{R}(J) \times \mathbb{R}(J) \times \mathbb{R}(J) \) by

\[
\bar{r}(t) = \begin{cases} 
1, & \text{if } t = r \\
0, & \text{if } t \neq r
\end{cases}
\]

The absolute value, \( |X| \) of \( X \in \mathbb{R}(J) \) is defined by

\[
|X|(t) = \begin{cases} 
\max \{ X(t), X(-t) \}, & \text{if } t \geq 0; \\
0, & \text{if } t < 0
\end{cases}
\]

Define a mapping \( \bar{d} : \mathbb{R}(J) \times \mathbb{R}(J) \times \mathbb{R}(J) \rightarrow \mathbb{R}^+ \cup \{0\} \) by

\[
\bar{d}(X,Y) = \sup_{0 \leq \alpha \leq 1} \overline{d}([X]^\alpha,[Y]^\alpha).
\]

It is known that \( (\mathbb{R}(J), \bar{d}) \) is a complete metric space.

2.9. **Definition.** A metric on \( \mathbb{R}(J) \) is said to be translation invariant if \( \bar{d}(X + Y, Y + Z) = \bar{d}(X, Z) \), for \( X, Y, Z \in \mathbb{R}(J) \).

2.10. **Definition.** Let \((X, \mu, \eta, *, \delta)\) be an IFMS. Then class of sequence space \( X = (X_{mnk}) \) of fuzzy numbers is said to be convergent to a fuzzy number \( X_0 \) if for every \( \varepsilon > 0 \) and \( t > 0 \), there exists a positive integer \( n_0 \) such that \( \bar{d}(\mu(X_{mnk},X_0,t)) \geq \varepsilon \) or \( \bar{d}(\eta(X_{mnk},X_0,t)) \leq \varepsilon \) for all \( m,n,k \geq n_0 \).

2.11. **Definition.** Let \((X, \mu, \eta, *, \delta)\) be an IFMS. Then class of sequence space \( X = (X_{mnk}) \) of fuzzy numbers is said to be (i) \( I \)-convergent to a fuzzy number \( X_0 \) if for each \( \varepsilon > 0 \) and \( t > 0 \) such that

\[
A = \{ m,n,k \in \mathbb{N} : \bar{d}(\mu(X_{mnk},X_0,t)) \geq \varepsilon \} \text{ in } I \text{ or }
A = \{ m,n,k \in \mathbb{N} : \bar{d}(\eta(X_{mnk},X_0,t)) \leq \varepsilon \} \text{ in } I
\]

The fuzzy number \( X_0 \) is called \( I \)-limit of the sequence \((X_{mnk})\) of fuzzy numbers and we write \( I - \lim X_{mnk} = X_0 \). (ii) \( I \)-bounded if there exists \( M > 0 \) such that

\[
\{ m,n,k \in \mathbb{N} : \bar{d}(X_{mnk},\bar{0}) > M \} \in I.
\]

2.12. **Definition.** Let \((X, \mu, \eta, *, \delta)\) be an IFMS. Then class of sequence space \( E_F \) of fuzzy numbers is said to be (i) solid (or normal) if \( (Y_{mnk}) \in E_F \) whenever \( (X_{mnk}) \in E_F \) and \( \bar{d}(Y_{mnk},\bar{0}) \leq \bar{d}(X_{mnk},\bar{0}) \) for all \( m,n,k \in \mathbb{N} \). (ii) symmetric if \( (X_{mnk}) \in E_F \) implies \( (X_{\pi(mnk)}) \in E_F \) where \( \pi \) is a permutation of \( \mathbb{N} \times \mathbb{N} \times \mathbb{N} \).

Let \( K = \{ m_1n_1k_1 < m_2n_2k_2 < ... \} \subseteq \mathbb{N} \) and \( E \) be a sequence space. A \( K \)-step space of \( E \) is a sequence space.
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\[ \lambda_{mnk}^E = \{ (X_{m,n,k}) \in w^3 : (m,n,k) \in E \} . \]

A canonical preimage of a sequence \( \{ (X_{m,n,k}) \} \in \lambda_{mnk}^E \) is a sequence \( \{ Y_{mnk} \} \in w^3 \) defined as

\[ Y_{mnk} = \begin{cases} X_{mnk}, & \text{if } m, n, k \in E \\ 0, & \text{otherwise.} \end{cases} \]

A canonical preimage of a step space \( \lambda_{mnk}^E \) is a set of canonical preimages of all elements in \( \lambda_{mnk}^E \), i.e. \( Y \) is in canonical preimage of \( \lambda_{mnk}^E \) if and only if \( Y \) is canonical preimage of some \( x \in \lambda_{mnk}^E \).

2.13. Definition. Let \((X, \mu, \eta, *, \delta)\) be an IFMS. Then class of sequence space \( E_F \) is said to be monotone if \( E_F \) contains the canonical pre-images of all its step spaces.

The following well-known inequality will be used throughout the article. Let \( p = (p_{mnk}) \) be any sequence of positive real numbers with \( 0 \leq p_{mnk} \leq \sup p_{mnk}p_{mnk} = G, D = \max \{1, 2G - 1\} \) then

\[ |a_{mnk} + b_{mnk}|^{p_{mnk}} \leq D (|a_{mnk}|^{p_{mnk}} + |b_{mnk}|^{p_{mnk}}) \]

for all \( m,n,k \in \mathbb{N} \) and \( a_{mnk}, b_{mnk} \in \mathbb{C} \).

Also \( |a_{mnk}|^{p_{mnk}} \leq \max \{1, |a|^G\} \) for all \( a \in \mathbb{C} \).

First we procure some known results; those will help in establishing the results of this article.

2.14. Lemma. Let \((X, \mu, \eta, *, \delta)\) be an IFMS. Then class of sequence space \( E_F \) is normal implies \( E_F \) is monotone. (For the crisp set case, one may refer to Kamthan and Gupta [26, page 53].)

2.15. Lemma. (Kostyrko et al., [30], Lemma 5.1). If \( I \subset 2^{\mathbb{N} \times \mathbb{N} \times \mathbb{N}} \) is a maximal ideal, then for each \( A \subset \mathbb{N} \) we have either \( A \in I \) or \( \mathbb{N} - A \in I \).

2.16. Definition. Let \( d \) be a mapping from \( R(I) \times R(I) \times R(I) \) into \( R^\ast(I) \times R^\ast(I) \times R^\ast(I) \) and let the mappings \( L, f : [0,1] \times [0,1] \to [0,1] \) be symmetric, non-decreasing Musielak Orlicz in both arguments and satisfy \( L \times L \times L(0,0,0) = 0 \) and \( f \times f \times f(1,1,1) = 1 \). Denote \( [d(X,Y,Z)]_{\alpha} = [\lambda_{\alpha}(X,Y,Z), (X,Y,Z)] \), for \( X,Y \in R(I) \times R(I) \times R(I) \) and \( 0 < \alpha < 1 \).

The \( (R(I) \times R(I) \times R(I), d, L \times L \times L, f \times f \times f) \) is called a fuzzy \( p- \) metric space and \( d \) a fuzzy translation metric, if

1. \( d(X,Y) = 0 \) if and only if \( X = Y = Z \),
2. \( d(X,Y) = d(Y,Z) d(Z,X) \) for all \( X,Y,Z \in X \),

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(3) for all \( X, Y, Z \in \mathbb{R}(I) \times \mathbb{R}(I) \times \mathbb{R}(I) \),
(i) \( d(X, Y, Z) (s + t + u) \geq L \times L \times L (d(X, Y) (s), d(Y, Z) (t), d(Z, X) (u)) \) whenever \( s \leq \lambda_1(X, Y), t \leq \lambda_1(Y, Z), u \leq \lambda_1(Z, X) \) and \( (s + t + u) \leq \lambda_1(X, Y, Z) \),
(ii) \( d(X, Y) (s + t + u) \leq f \times f \times f (d(X, Y) (s), d(Y, Z) (t), d(Z, X) (u)) \) whenever \( s \geq \lambda_1(X, Y), t \geq \lambda_1(Y, Z), u \geq \lambda_1(Z, X) \) and \( (s + t + u) \leq \lambda_1(X, Y, Z) \).

3. Some new Intuitionistic sequence spaces of fuzzy numbers

The main aim of this article to introduce the following sequence spaces and examine topological and algebraic properties of the resulting sequence spaces. Let \((X, \mu, \eta, *, \delta)\) be an IFMS and \(p = (p_{mnk})\) be a sequence of positive real numbers for all \(m, n, k \in \mathbb{N}\).

\(f = (f_{mnk})\) be a Musielak-Orlicz function, \((X, \|d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p)\) be a fuzzy \(p\)-metric space, and \(\alpha_{mnk}(x) = \mu(X^{1/m+m+n+k}, 0, t)\) be a sequence of fuzzy numbers. Using the concept of fuzzy metric, we introduce the following class of sequence:

\[
\left\{ \begin{array}{l}
\Gamma^{3RF}_{f(\mu,0)}, \|d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p = \\
\{ (m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \|\alpha_{mnk}(x), d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p \leq 1 - \epsilon \} \in I.
\end{array} \right.
\]

3.1. Theorem. Let \(f = (f_{mnk})\) be a Musielak-Orlicz function, \((X, \mu, \eta, *, \delta)\) be an IFMS. Then class of sequence space \(\Gamma^{3RF}_{f(\mu,0)}, \|d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p\) is a linear space.

Proof: It is trivial. Therefore omit the proof.

3.2. Theorem. Let \((X, \mu, \eta, *, \delta)\) be an IFMS. Then class of sequence space \(\Gamma^{3RF}_{f(\mu,0)}, \|d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p\) is solid and as such monotone

Proof: Consider two sequences \((X_{mnk})\) and \((Y_{mnk})\) such that \(|X_{mnk}| \leq |Y_{mnk}|\), for all \(m, n, k \in \mathbb{N}\) [see Definition 2.12] and \(Y_{mnk} \in \Gamma^{3RF}_{f(\mu,0)}, \|d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p\).

We have \(\alpha_{mnk}(X) < \alpha_{mnk}(Y) \to 0\), as \(m, n, k \to \infty\).
\[
\Rightarrow \alpha_{mnk}(X) \in \Gamma^{3RF}_{f(\mu,0)}, \|d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p.
\]
Thus the class \(\Gamma^{3RF}_{f(\mu,0)}, \|d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p\) is solid. The class of sequences
\[
\Gamma^{3RF}_{f(\mu,0)}, \|d(x_1), d(x_2), \ldots, d(x_{n-1})\|_p
\]
is monotone follows from the Lemma 2.15.
3.3. **Theorem.** Let \( (X, \mu, \eta, *, \delta) \) be an IFMS. Then class of sequence space \( \left[ \Gamma_{f(\mu, \eta)}^{3(F)} \right. \| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \|_p \left. \right] \) is not convergence free.

**Proof:** Consider a sequence \( (X_{mnk}) \in \left[ \Gamma_{f(\mu, \eta)}^{3(F)} \right. \| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \|_p \left. \right] \) defined as follows: For \( m, n, k \) are even

\[
\alpha_{mnk}(X) = \begin{cases} 
1 + (mnk)^3 \Gamma, & \text{for} - (mnk)^{-3} \leq t \leq 0, \\
1 - (mnk)^3 \Gamma, & \text{for} 0 \leq t \leq (mnk)^{-3}, \\
0, & \text{otherwise}
\end{cases}
\]

and for \( m, n, k \) are odd, \( \alpha_{mnk}(X) = \bar{0} \).

Now for \( \alpha \in (0, 1] \),

\[
\alpha_{mnk}(X)^\gamma = \begin{cases} 
(\gamma - 1) (mnk)^{-3}, & \text{for} \ m, n, k \text{ even} \\
[0, 0], & \text{for} \ m, n, k \text{ odd}
\end{cases}
\]

Then \( \alpha_{mnk}(X) = (\gamma - 1) (mnk)^{-3} \to \bar{0} \), as \( m, n, k \to \infty \), and \( \alpha_{mnk}(X) = (1 - \gamma) (mnk)^{-3} \to \bar{0} \), as \( m, n, k \to \infty \). Thus,

\( X_{mnk} \in \left[ \Gamma_{f(\mu, \eta)}^{3(F)} \right. \| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \|_p \left. \right] \). Let us define a sequence \( (Y_{mnk}) \) as follows For \( m, n, k \) are even

\[
\alpha_{mnk}(Y) = \begin{cases} 
1 + (mnk)^4 \Gamma, & \text{for} - (mn)^{-1} \leq t \leq 0, \\
1 - (mnk)^4 \Gamma, & \text{for} 0 \leq t \leq (mnk)^{-1}, \\
0, & \text{otherwise}
\end{cases}
\]

and for \( m, n, k \) are odd, \( \alpha_{mnk}(Y) = \bar{0} \).

Now for \( \alpha \in (0, 1] \),

\[
\alpha_{mnk}(Y)^\gamma = \begin{cases} 
(\gamma - 1) (mnk)^{-1}, (1 - \gamma) (mnk)^{-1}, & \text{for} \ m, n, k \text{ even} \\
[0, 0], & \text{for} \ m, n, k \text{ odd}
\end{cases}
\]

Then \( \alpha_{mnk}(Y) = (\gamma - 1) (mnk)^{-3} \neq \bar{0} \), as \( m, n, k \to \infty \), and \( \alpha_{mnk}(Y) = (1 - \gamma) (mnk)^{-3} \neq \bar{0} \), as \( m, n, k \to \infty \). Thus,

\( Y_{mnk} \notin \left[ \Gamma_{f(\mu, \eta)}^{3(F)} \right. \| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \|_p \left. \right] \). Hence

\( \left[ \Gamma_{f(\mu, \eta)}^{3(F)} \right. \| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \|_p \left. \right] \) is not convergence free.

3.4. **Theorem.** Let \( (X, \mu, \eta, *, \delta) \) be an IFMS. Then class of sequence space \( \left[ \Gamma_{f(\mu, \eta)}^{3(F)} \right. \| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \|_p \left. \right] \) is symmetric

**Proof:** Let \( (X_{mnk}) \in \left[ \Gamma_{f(\mu, \eta)}^{3(F)} \right. \| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \|_p \left. \right] \). Let \( (Y_{mnk}) \) be a arrangement of the sequence \( (X_{mnk}) \) such that \( X_{mnk} = Y_{pmqtrk} \) for each \( m, n, k \in \mathbb{N} \). Then \( \alpha_{mnk}(X) = \alpha_{mnk}(Y) \), as \( m, n, k \to \infty \), and \( \alpha_{mnk}(X) = \alpha_{mnk}(Y) \to \bar{0} \), as
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Thus, \((Y_{mnk}) \in \Gamma_{3I}(F)\),
\[
\left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p = \left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p,
\]

3.5. Theorem. Let \((X, \mu, \eta, *, \delta)\) be an IFMS. Then class of sequence space

\[
\left[ \Gamma_{3I}(F) \right]_{\left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p}
\]

is a sequence algebra.

Proof: Let \((X_{mnk}), (Y_{mnk}) \in \left[ \Gamma_{3I}(F) \right]_{\left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p}\), then we have \(\alpha_{mnk}(X) \to 0\), as \(m, n, k \to \infty\) and \(\alpha_{mnk}(Y) \to 0\), as \(m, n, k \to \infty\). The result follows from the following inequalities

\[
\alpha_{mnk}(X \otimes Y) \leq \alpha_{mnk}(X) + \alpha_{mnk}(Y) \to 0,
\]

Hence the class \(\left[ \Gamma_{3I}(F) \right]_{\left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p}\) is sequence algebra.

3.6. Theorem. (i) Let \((X, \mu, \eta, *, \delta)\) be an IFMS. Then class of sequence space \((r_{mnk})\) satisfies \(\Delta_2\)– condition, then

\[
\left[ \Gamma_{3I}(F) \right]_{\left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p} = \left[ \Gamma_{3I}(F) \right]_{\left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p}.
\]

(ii) Let \((X, \mu, \eta, *, \delta)\) be an IFMS. Then class of sequence space \((s_{mnk})\) satisfies \(\Delta_2\)– condition, then

\[
\left[ \Gamma_{3I}(F) \right]_{\left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p} = \left[ \Gamma_{3I}(F) \right]_{\left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p}.
\]

Proof: Let \((X, \mu, \eta, *, \delta)\) be an IFMS. Then class of sequence space \((r_{mnk})\) satisfies \(\Delta_2\)– condition, we get

\[
\left[ \Gamma_{3I}(F) \right]_{\left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p} \subset \left[ \Gamma_{3I}(F) \right]_{\left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p}.
\]

To prove the inclusion

\[
\left[ \Gamma_{3I}(F) \right]_{\left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p} \subset \left[ \Gamma_{3I}(F) \right]_{\left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p}.
\]

let \(a \in \left[ \Gamma_{3I}(F) \right]_{\left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p}\). Then for all \(\alpha_{mnk}(X)\) with \(\alpha_{mnk}(X) \in \left[ \Gamma_{3I}(F) \right]_{\left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p}\). We have

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |x_{mnk}a_{mnk}| < \infty.
\]

Since \((X, \mu, \eta, *, \delta)\) be an IFMS. Then class of sequence space \((r_{mnk})\) satisfies \(\Delta_2\)– condition, then

\[
\alpha_{mnk}(Y) \in \left[ \Gamma_{3I}(F) \right]_{\left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p},
\]

we get \(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |y_{mnk}a_{mnk}| < \infty\). by (3.2). Thus

\[
(a_{mnk}) \in \left[ \Gamma_{3I}(F) \right]_{\left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p} = \left[ \Gamma_{3I}(F) \right]_{\left\| (d(x_1), d(x_2), \cdots, d(x_{n-1})) \right\|_p}.
\]
and hence
\[(a_{mnk}) \in \left[ \Gamma^{3I(F)}_{f(\mu,\eta,r)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \right].\] This gives that
\[(3.3) \quad \Gamma^{3I(F)}_{f(\mu,\eta,r)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \subseteq \Gamma^{3I(F)}_{f(\mu,\eta,r)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\]
we are granted with (3.1) and (3.3)
\[(3.3) \quad \Gamma^{3I(F)}_{f(\mu,\eta,r)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \subseteq \Gamma^{3I(F)}_{f(\mu,\eta,r)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\]
(ii) Similarly, one can prove that
\[(3.3) \quad \Gamma^{3I(F)}_{f(\mu,\eta,r)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p \subseteq \Gamma^{3I(F)}_{f(\mu,\eta,r)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\]
if the sequence \((s_{mnk})\) satisfies \(\Delta_2\) condition.

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