NEW SETS AND TOPOLOGIES IN IDEAL TOPOLOGICAL SPACES

J. Sebastian Lawrence 
Department of Mathematics  
Sathyabama University, Chennai-600119 
Email: lawrence.seba@gmail.com

R. Manoharan, 
Department of Mathematics, 
Sathyabama University, Chennai-600119. 
Email: mano_rl@yahoo.co.in

ABSTRACT

In this paper, in an ideal space \((X, \tau, \mathcal{I})\), we introduce and study about a new topology called \(R_{IC}\)-topology which is finer than \(\tau\). Also we analyze the relationships with the already existing topologies.

AMS Subject classification: 54A05, 54A10

Keywords- \(R_p\)-perfect sets, \(R_{IC}\)-open sets, \(R_c\)-open sets, \(R_{IC}\)-topology.

1. Introduction

A nonempty collection of subsets of a set \(X\) is said to be an ideal on \(X\), if it satisfies the following two conditions:

(i) if \(A \in \mathcal{I}\) and \(B \subseteq A\), then \(B \in \mathcal{I}\) (heredity);
(ii) if \(A \in \mathcal{I}\) and \(B \in \mathcal{I}\), then \(A \cup B \in \mathcal{I}\) (finite additivity).

An ideal topological space (or ideal space) \((X, \tau, \mathcal{I})\) means a topological space \((X, \tau)\) with an ideal \(\mathcal{I}\) defined on \(X\). For a given subset \(A\) of \(X\), \(\text{cl}(A)\) and \(\text{int}(A)\) are used to denote the closure of \(A\) and interior of \(A\), respectively, with respect to the topology.

For a given point \(x\) in a space \((X, \tau)\), the system of open neighborhoods of \(x\) is denoted by \(N(x) = \{U \subseteq X : x \in U\}\). Let \((X, \tau, \mathcal{I})\) be a topological space with an ideal \(\mathcal{I}\) defined on \(X\), then for any subset \(A\) of \(X\), \(A^*(\mathcal{I}, \tau) = \{x \in X/A \cap U \in \mathcal{I} \text{ for every } U \in N(x)\}\) is called the local function of \(A\) with respect to \(\mathcal{I}\) and \(\tau\). If there is no ambiguity, we will write \(A^*(\mathcal{I})\) or simply \(A^*\) for \(A^*(\mathcal{I}, \tau)\). Also, \(c(\mathcal{I}) = A \cup A^*\) defines a Kuratowski closure operator[3] for the topology \(\tau^*(\mathcal{I}, \tau)\) (or simply \(\tau^*\)) which is finer than \(\tau\). Jankovic and Hamlett[1, 3, 4] made an extensive study on topological ideals which initiated the generalization of many topological properties.

Manoharanet al. [7] introduced \(R^*\)-topology which is finer than \(\tau^*\). Let \((X, \tau, \mathcal{I})\) be an ideal topological space. A subset \(A\) of \(X\) is said to be \(R^*\)-perfect if \(A \setminus A \in \mathcal{I}\). The collection of all perfect sets satisfies the conditions of being a basis for some topology and it will be called as \(R^*(\tau, \mathcal{I})\). We define \(R^*(\tau, \mathcal{I}) = \{A \subseteq X / X \setminus A \in R^*(\tau, \mathcal{I})\}\) on a nonempty set. Clearly, \(R^*(\tau, \mathcal{I})\) is a topology if the set \(X\) is finite. The members of the collection \(R^*(\tau, \mathcal{I})\) will be called \(R^*\)-open sets. If there is no confusion about the topology \(\tau\) and the ideal \(\mathcal{I}\), then \(R^*(\tau, \mathcal{I})\) can be called as \(R^*\)-topology when \(X\) is finite. For a given subset \(A\) of a space \((X, \tau)\), \(c(\mathcal{I})\) and \(\text{int}(\mathcal{I})\) are used to denote the closure of \(A\) and interior of \(A\), respectively, with respect to the topology.

A perfect set in topological spaces is a set without isolated points (dense in itself) and closed. Hayashi introduced *-perfect sets[2] in ideal topological spaces. A subset \(A\) of an ideal space is *-perfect if \(A = A^*\). Later, Manoharanet al. [7] introduced \(R^*_p\)-perfect sets. Mohammad Shabir et al. [8] introduced and studied soft topological spaces. Kandil et al. [6] extended soft topological properties to soft ideal topological spaces. Recently, Rodyna A. Hosny [9] extended the idea of perfect sets to soft ideal topological spaces. In this chapter, we introduce \(L_p\)-perfect sets, \(R_p\)-perfect sets and \(C_p\)-perfect sets, study their properties and obtain some relation with other sets in ideal topological spaces which also can be extended to soft ideal topological spaces.
2. L₁-Perfect, R₁-Perfect and C₁-perfect sets

In this section, we define three collections of subsets \( \mathcal{L}_1, \mathcal{R}_1 \) and \( \mathcal{C}_1 \) in an ideal space and study some of their properties.

Definition 2.1. Let \( (X, \tau, \mathcal{I}) \) be an ideal topological space. A subset \( A \) of \( X \) is said to be

(i) L₁-perfect if \( A \setminus A' \in \mathcal{I} \)
(ii) R₁-perfect if \( A \setminus A \in \mathcal{I} \) and
(iii) C₁-perfect if \( A \) is both L₁-perfect and R₁-perfect.

The collection of L₁-perfect sets, R₁-perfect sets and C₁-perfect sets in \( (X, \tau, \mathcal{I}) \) are denoted by \( \mathcal{L}_1, \mathcal{R}_1 \) and \( \mathcal{C}_1 \) respectively.

Remark 2.2. If \( \mathcal{I} = \{\emptyset\} \), then \( \mathcal{L}_1 \) is the collection of all dense-in-itself sets, \( \mathcal{R}_1 \) is the collection of all closed sets and \( \mathcal{C}_1 \) is the collection of all perfect sets.

Remark 2.3. Every perfect set is both R₁-perfect and L₁-perfect (and hence C₁-perfect). The following examples shows that every L₁-perfect and R₁-perfect sets need not be a perfect set.

Example 2.4. Let \( X = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\} \) and \( I = \{\emptyset, \{a\}, \{c\}, \{b, c\}\} \). The sets \( \{b\} \) and \( \{c\} \) are R₁-perfect and L₁-perfect sets but not perfect sets.

Proposition 2.5. Let \( (X, \tau, \mathcal{I}) \) be an ideal space. Let \( A \) be a subset of \( X \).

(i) If \( A \) is L₁-perfect, then it is L₁-perfect.
(ii) If \( A \) is R₁-perfect, then it is R₁-perfect.

The following example shows that the converse of the above results need not be true.

Example 2.6. Let \( X = \{a, b, c, d\}, \tau = \{\emptyset, \{c\}, \{a, b\}, \{a, b, c\}, X\} \) and \( I = \{\emptyset, \{a\}\} \). The set \( \{b, d\} \) is L₁-perfect but not L₁-perfect set. The set \( \{a, d\} \) is R₁-perfect but not R₁-perfect.

Proposition 2.7. Let \( (X, \tau, \mathcal{I}) \) be an ideal space. Let \( A \) and \( B \) are two subsets of \( X \).

(i) If \( A \) and \( B \) are R₁-perfect sets, then \( A \cup B \) is an R₁-perfect set.
(ii) If \( A \) and \( B \) are L₁-perfect sets, then \( A \cup B \) is an L₁-perfect set.

Proof.

Let \( A \) and \( B \) are R₁-perfect sets. Then \( A \setminus A \in \mathcal{I} \) and \( B \setminus B \in \mathcal{I} \). By finite additive property of ideals, \( (A \setminus A) \cup (B \setminus B) \in \mathcal{I} \). Since \( (A \setminus A) \cup (B \setminus B) \in \mathcal{I} \), by heredity property, \( (A \setminus A) \cup (B \setminus B) \in \mathcal{I} \). Hence \( (A \setminus A) \cup (B \setminus B) \in \mathcal{I} \). This proves (i).

Let \( A \) and \( B \) are L₁-perfect sets. Since \( A \) and \( B \) are L₁-perfect sets, \( A \setminus A \in \mathcal{I} \) and \( B \setminus B \in \mathcal{I} \). Hence, by finite additive property of ideals, \( (A \setminus A) \cup (B \setminus B) \in \mathcal{I} \).

Since \( (A \setminus A) \cup (B \setminus B) \in \mathcal{I} \), by heredity property, \( (A \setminus A) \cup (B \setminus B) \in \mathcal{I} \). This proves \( A \cup B \) is an L₁-perfect set.

Corollary 2.8. In an ideal space \( (X, \tau, \mathcal{I}) \),

(i) Finite union of R₁-perfect sets is an R₁-perfect set.
(ii) Finite union of L₁-perfect sets is an L₁-perfect sets.

Proposition 2.9. If \( A \) and \( B \) are R₁-perfect sets, then \( A \cap B \) is an R₁-perfect set.

Proof.

Suppose \( A \) and \( B \) are R₁-perfect sets. Then \( A \setminus A \in \mathcal{I} \) and \( B \setminus B \in \mathcal{I} \). By the finite additive property of ideals, \( (A \setminus A) \cup (B \setminus B) \in \mathcal{I} \). Since \( (A \setminus A) \cup (B \setminus B) \in \mathcal{I} \), by heredity property, \( (A \setminus A) \cup (B \setminus B) \in \mathcal{I} \). Also \( (A \setminus A) \cup (B \setminus B) \in \mathcal{I} \). This proves the result.

Corollary 2.10. Finite intersection of R₁-perfect sets is an R₁-perfect set.
Proposition 2.11. Let $A$ be a subset of an ideal space $(X, \tau, \mathcal{I})$. Then
(i) If $A$ is $C$-perfect, then $A^{\Delta} A' \in \mathcal{I}$.
(ii) If $A$ is $\tau$-closed, then $A$ is $R_1$-perfect.
(iii) If $A$ is dense-in-itself, then $A$ is $I_4$-perfect.

Corollary 2.12. For any set $A$, $A^{\ast}$ and $cl(A)$ are $R_1$-perfect sets.

Corollary 2.13. If a set $A$ is an $R_1$-perfect set, then $cl(A)$ is also an $R_1$-perfect.

Proof.
Since $cl(A) = A \cup A^{\ast}$ and both $A$ and $A^{\ast}$ are $R_1$-perfect sets, $cl(A)$ is also an $R_1$-perfect set.

Corollary 2.14. If $A$ is an $R_1$-perfect set, then $A \cap A^{\ast}$ is also an $R_1$-perfect.

Proposition 2.15. Let $(X, \tau, \mathcal{I})$ be an ideal space. Let $A$ and $B$ be two subsets of $X$ such that $A \subseteq B$ and $A^{\ast} = B^{\ast}$.
Then
(i) $B$ is $R_1$-perfect if $A$ is $R_1$-perfect.
(ii) $A$ is $I_4$-perfect if $B$ is $I_4$-perfect.

Proof.
Let $A$ is $R_1$-perfect. Then $A \setminus A \subseteq I$. Now, $B \setminus B = A^{\ast} \setminus B \subseteq A^{\ast} \setminus A$. By heredity property of ideals, $B \setminus B \in \mathcal{I}$. Then $B$ is $R_1$-perfect.
Let $B$ is $I_4$-perfect. Then $B \setminus B^{\ast} \subseteq \mathcal{I}$. Now, $A \setminus A^{\ast} = A \setminus B^{\ast} \subseteq B \setminus B^{\ast}$. By heredity property of ideals, $A \setminus A^{\ast} \in \mathcal{I}$.
Hence $A$ is $I_4$-perfect.

Proposition 2.16. Let $(X, \tau, \mathcal{I})$ be an ideal topological space with $X$ is finite. Then the collection $R_0$ of all $R_1$-perfect sets is a topology which is finer than the collection of all $\tau$-closed sets.

Proof.
$X$ and $\emptyset$ are $R_1$-perfect sets. By Corollary 2.8, finite union of $R_1$-perfect sets is an $R_1$-perfect set and by Corollary 2.10, finite intersection of $R_1$-perfect sets is $R_1$-perfect. Hence the collection $R_1$ is a topology if $X$ is finite. Also, by Proposition 2.11 every $\tau$-closed set is an $R_1$-perfect set. Hence the topology $R_1$ is finer than the collection of all $\tau$-closed sets if $X$ is finite.

3. $R_{IC}$-Topology

By Proposition 2.9 and 2.11, we observe that the collection of all $R_1$-perfect sets $R_1=\{A \subseteq X / A^{\ast} \setminus A \}$ satisfies the conditions of being a basis for some topology. We will denote the topology as $R_{IC}(\mathcal{I}, \tau)$. If there is no confusion about the topology $\tau$ and the ideal $\mathcal{I}$, we will call the topological space $R_{IC}(\mathcal{I}, \tau)$ as $R_{IC}$. The members of $R_{IC}(\mathcal{I}, \tau)$ are called $R_{IC}$-open sets and the complements of $R_{IC}$-open sets are called $R_{IC}$-closed sets.

Example 3.1. Let $X=\{a, b, c, d\}, \tau=\{\emptyset, \{d\}, \{a, c\}, \{a, c, d\}\}$ and $I=\{\emptyset, \{c\}, \{d\}, \{c, d\}\}$. Then $\{\emptyset, \{a\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X\}$ is $R_{IC}$-topology.

Remark 3.2. In an ideal space $(X, \tau, \mathcal{I})$, every $\tau$-closed set is an $R_{IC}$-open whereas every open set is an $R_{IC}$-closed set.

Definition 3.3. Let $A$ be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. We define $R_{IC}$-interior of the set $A$ is the largest $R_{IC}$-open set contained in $A$. We will denote $R_{IC}$-interior of a set $A$ by $R_{IC}$$\text{int}(A)$.

Definition 3.4. Let $A$ be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. A point $x \in A$ is said to be an $R_{IC}$-interior point of the set $A$ if there exists an $R_{IC}$-open set $U$ of $x$ such that $x \in U \subseteq A$.

Definition 3.5. Let $(X, \tau, \mathcal{I})$ be an ideal space and $x \in X$. We define $R_{IC}$-neighborhood of $x$ as an $R_{IC}$-open set containing $x$. We denote the set of all $R_{IC}$-neighborhoods of $x$ by $R_{IC}\text{-N}(x)$. 

95
Definition 3.6. Let A be a subset of an ideal topological space \((X, \tau, \mathfrak{F})\). We define \(R_{IC}\)-closure of the set A as the smallest \(R_{IC}\)-closed set containing A. We will denote \(R_{IC}\)-closure of a set A by \(R_{IC}\text{-cl}(A)\).

Proposition 3.7. In an ideal space \((X, \tau, \mathfrak{F})\), \(R_{IC}\)-topology is coarser than \(R_{IC}\)-topology.

Proof: Since every \(R\)-perfect is an \(R^*\)-perfect set, \(R_{IC}\)-topology is coarser than \(R_{IC}\)-topology.

Example 3.8. Let \(X = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, b, d\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}, X\}\) and \(I = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}\). The set \(\{\emptyset, \{c\}, \{d\}, \{c, d\}, \{a, d\}, \{b, d\}, \{b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}, X\}\) is \(R_{IC}\)-topology.

Theorem 3.9. Let \(A\) and \(B\) be subsets of an ideal space \((X, \tau, \mathfrak{F})\). Then the following properties hold:
(i) \(R_{IC}\)-cl(A) = \(\{F : A \subseteq F\text{ and } F\text{ is }R_{IC}\text{-closed set}\}\).
(ii) \(R_{IC}\)-cl(A) is the largest \(R_{IC}\)-closed set of X contained in A.
(iii) \(A\) is \(R_{IC}\)-open if and only if \(A = R_{IC}\)-cl(A).
(iv) \(R_{IC}\text{-int}(R_{IC}\text{-cl}(A)) = R_{IC}\text{-cl}(R_{IC}\text{-int}(A))\).
(v) If \(A \subseteq B\), then \(R_{IC}\text{-cl}(A) \subseteq R_{IC}\text{-cl}(B)\).

Definition 3.10. Let A be a subset of an ideal topological space \((X, \tau, \mathfrak{F})\). We define \(R_{IC}\)-closure of the set A as the smallest \(R_{IC}\)-closed set containing A. We will denote \(R_{IC}\)-closure of a set A by \(R_{IC}\text{-cl}(A)\).

Remark 3.11. For any subset A of an ideal topological space \((X, \tau, \mathfrak{F})\), \(R_{IC}\text{-cl}(A) \subseteq cl(A)\). If \(\mathfrak{F} = \{\emptyset\}\) then \(R_{IC}\text{-cl}(A) = cl(A)\).

Theorem 3.12. Let \(A\) and \(B\) be subsets of an ideal space \((X, \tau, \mathfrak{F})\) with X finite. Then the following properties hold:
(i) \(R_{IC}\text{-int}(X-A) = X \setminus (R_{IC}\text{-cl}(A))\).
(ii) \(R_{IC}\text{-cl}(X-A) = X \setminus (R_{IC}\text{-int}(A))\).

Definition 3.14. Let \((X, \tau, \mathfrak{F})\) be an ideal space with X finite. The collection of all \(R_{IC}\)-closed sets is a topology which is denoted by \(R_{IC}\text{-topology}\). If there is no confusion about the topology \(\tau\) and the ideal \(\mathfrak{F}\), we will call the topological space \((X, \tau, \mathfrak{F})\) as \(R_{IC}\text{-topology}\). The members of \(R_{IC}\text{-topology}\) are called \(R_{IC}\text{-open sets}\) and the complements of \(R_{IC}\text{-open sets}\) are called \(R_{IC}\text{-closed sets}\).

Definition 3.15. Let A be a subset of an ideal topological space \((X, \tau, \mathfrak{F})\). We define \(R_{IC}\)-interior of the set A is the largest \(R_{IC}\)-open set contained in A. We will denote \(R_{IC}\)-interior of a set A by \(R_{IC}\text{-int}(A)\).

Remark 3.16. Every open set is \(R_{IC}\)-open and every \(R_{IC}\)-open set is \(R^*\)-open.

Remark 3.17. In an ideal space \((X, \tau, \mathfrak{F})\), a subset A of X is \(R_{IC}\)-open set if and only if \(A^\circ\) is an \(R_{IC}\)-open set. In other words, a subset A of X is \(R_{IC}\)-closed set if and only if \(A^\circ\) is an \(R_{IC}\)-open set.
Example 3.18. (for $R_I$-topology) Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a, b\}, X\}$ and $I = \{\emptyset, \{a\}\}$. Then the set 
$\{\emptyset, \{c, d\}, \{b, c, d\}, X\}$ is $R_I$-topology.

Proposition 3.19. In an ideal space $(X, \tau, J)$, $\tau \subseteq R_I \subseteq R^*$.

Remark 3.20. The following example shows that $R^* \neq R_I \neq \tau$.

Example 3.21. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{a, b, c\}, X\}$ and $I = \{\emptyset, \{a\}, \{a, b\}\}$. Then the collection $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$ is $R^*$-topology.

The collection $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}, X\}$ is $R^*$-topology.

Remark 3.22. We observe that $\tau \subseteq R^* \subseteq \tau^*$.

References:


