Study On Fuzzy $\gamma^*$—fuzzy semi open sets and Fuzzy $\gamma^*$—semi closed set in fuzzy topological spaces

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Abstract

In this paper, we introduce a new classes of sets called fuzzy $\gamma^*$-semi open sets and fuzzy $\gamma^*$-semi closed sets and its properties are established in fuzzy topological spaces.

Key Words and Phrases: Fuzzy $\gamma$-open, fuzzy $\gamma$-closed, fuzzy $\gamma$-semi open, fuzzy $\gamma$-closed, fuzzy $\gamma$-semi interior, fuzzy $\gamma$-semi closure.

1 Introduction

The concept of fuzzy sets operations were first introduced by L.A. Zadeh [7] in this paper. Let $X$ be non empty set and $I$ be the unit interval $[0, 1]$. A fuzzy set mapping $X$ into $I$. In 1968 Chang [4] introduced the concept of fuzzy topological space. Azad introduced the notions of fuzzy semi open and fuzzy semi closed sets and T. Noiri and O.R Sayed [6] introduced the notion of $\gamma$ open sets and $\gamma$ closed sets. In this paper we introduce fuzzy $\gamma^*$ semi open sets, fuzzy $\gamma^*$-semi closed sets and its properties are established in fuzzy topological spaces.
2 Preliminaries

Through this paper \((X, \tau)\) and \((Y, \sigma)\) denote fuzzy topological spaces. For a fuzzy set \(A\) in a fuzzy topological space \(X\) or \(Y\). \(\text{cl}(A),\ \text{int}(A),\ A^c\) denote the closure, interior, complement of \(A\) respectively. By \(0_X\) and \(1_X\) we mean the constant fuzzy sets taking on the values 0 and 1 respectively.

**Definition 2.1 (Fuzzy Sets).** A fuzzy set \(X\) is a function with domain \(X\) and values in \(I\). That is an element of \(I^X\). The subset of \(X\) in which \(A\) assumes non-zero values is known as the support of \(A\) for every \(x \in X\), \(A(x)\) is called the grade of membership of \(x\) in \(A\). And \(X\) is called carrier of the fuzzy set \(A\). If \(A\) takes only 0 and 1, then \(A\) is a crisp set in \(X\).

**Operations of the fuzzy sets**

**Definition 2.2.** Let \(A, B \in I^X\) we define the following fuzzy sets:

1. (Union) \(A \lor B \in I^X\) by \((A \lor B)(x) = \max \{A(x), B(x)\}\) for every \(x \in X\).

2. (Intersection) \(A \land B \in I^X\) by \((A \land B)(x) = \min \{A(x), B(x)\}\) for every \(x \in X\).

3. (Complement) \(A^c \in I^X\) by \(A^c(x) = 1 - A(x)\) for every \(x \in X\).

4. Let \(f : X \to Y, A \in I^X\) and \(B \in I^Y\) then \(f(A)\) is a fuzzy set in \(Y\) defined by

\[
f(A)(y) = \begin{cases} \sup \{A(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq 0 \\ 0, & \text{if } f^{-1}(y) = 0 \end{cases}
\]

**Definition 2.3.** Let \(A \in I^X\) and \(B \in I^Y\) then by \(A \times B\) we denote the fuzzy set in \(X \times Y\) for each \((A \times B)(x, y) = \min \{A(x), B(y)\}\) for every \((x, y) \in X \times Y\).

**Proposition 2.4.** If \(A\) is a fuzzy set of \(X\) and \(B\) is a fuzzy set of \(Y\) then \(1 - (A \times B) = (A^c \times 1) \lor (1 \times B^c)\).
Proof. If \((1 - (A \times B)) (x, y) = \max (1 - A(x), 1 - B(y)) = \max [(A^c \times 1) (x, y), (1 \times B^c) (x, y)] = ((A^c \times 1) \lor (1 \times B^c)) (x, y)\), for each \((x, y) \in X \times Y\).

**Definition 2.5.** A family \(\tau \subseteq I^X\) of fuzzy sets is called a fuzzy topology for \(X\) if it satisfies the following three axioms.

1. \(\emptyset, \top \in \tau\)
2. \(\forall A, B \in \tau \implies A \land B \in \tau\)
3. \(\forall (A_j)_{j \in J} \in \tau \implies \bigvee_{j \in J} A_j \in \tau\). Then pair \((X, \tau)\) is called a fuzzy topological space (or) \(fts\) for short. The elements of \(\tau\) are called fuzzy open sets.

**Example 2.6.** Let \(X = \{a, b, c\}\) and \(\tau = \{0, 1, \{a, 1, b, 2, c, 3\}, \{a, 5, b, 1, c, 4\}, \{a, 1, b, 1, c, 3\}, \{a, 5, b, 2, c, 4\}\}\)

(i) \(0 \in \tau\) and \(1 \in \tau\).

(ii) \(\{a, 1, b, 2, c, 3\} \lor \{a, 1, b, 1, c, 3\} = \{a, 1, b, 2, c, 3\} \in \tau\)

(iii) \(\{a, 5, b, 1, c, 4\} \land \{a, 1, b, 1, c, 3\} = \{a, 1, b, 1, c, 3\} \in \tau\)

\(\therefore \tau\) is a fuzzy topology and the pair \((X, \tau)\) is called fuzzy topological space.

**Definition 2.7.** A fuzzy set \(A\) of \((X, \tau)\) is called

1. Fuzzy semi open (in short Fs open) if \(A \leq cl (Int (A))\) and a fuzzy semi closed (in short Fs – closed) if \(Int (cl (A)) \leq A\)
2. Fuzzy preopen (in short Fp-open if \(A \leq Int (cl (A))\) and a fuzzy pre closed (in short Fp-closed) if \(cl (Int (A)) \leq A\).

3. Fuzzy strongly semi open (in short Fss–open if \(A \leq int (cl (A))\) and a fuzzy strongly semi closed (in short Fss-closed) \(cl (Int (cl (A)) \leq A\).

4. Fuzzy \(\gamma\)-open if \(A \leq (int (cl (A))) \lor (cl (int A))\) & fuzzy \(\gamma\)-closed if \(cl (int (A)) \land int (cl (A)) \leq A\).

**Definition 2.8.** If \(\lambda\) is a fuzzy set of \(X\) and \(\mu\) is a fuzzy set of \(Y\), then \((\lambda \times \mu)(x, y) = \min [\lambda(x), \mu(y)]\), for each \(X \times Y\).
Lemma 2.9. Let $X$ and $Y$ be fuzzy topological spaces such that $X$ is product related to $Y$. Then for fuzzy set $A$ of $X$ and $B$ of $Y$.

1) $\text{cl} \ (A \times B) = \text{cl} (A) \times \text{cl} (B)$

2) $\text{int} \ (A \times B) = \text{int} (A) \times \text{int} (B)$.

Definition 2.10. For fuzzy sets $\lambda, \mu, \vartheta$ and $\omega$ in a set $S$ one has, $(\lambda \wedge \mu) \times (\vartheta \wedge \omega) = (\lambda \times \omega) \wedge (\mu \times \vartheta)$

Definition 2.11. Let $A$ be any fuzzy set in the fuzzy topological space $X$, then we define $\gamma - \text{cl} \ (A) = \wedge \{B : B \supseteq A, B \text{ is fuzzy } \gamma \text{-closed set}\}$ and $\gamma - \text{int} \ (A) = \vee \{B : B \subseteq A, B \text{ is fuzzy } \gamma \text{-open set}\}$.

Properties 2.12. Let $A$ be any fuzzy set in the fuzzy topological space $X$, then

a) $\gamma - \text{cl} \ (A^c) = (\gamma - \text{int} \ (A))^C$

b) $\gamma - \text{int} \ (A^c) = (\gamma - \text{cl} \ (A))^C$.

Properties 2.13. Let $A$ and $B$ be any two fuzzy sets in a fuzzy topological spaces of $X$, then

1) $\gamma - \text{int} \ (0) = 0, \gamma - \text{int} \ (1) = 1$.

2) $\gamma - \text{int} \ (A)$ is fuzzy $\gamma$-open in $X$.

3) $\gamma (\text{int} (\gamma \text{ int} \ (A))) = \gamma - \text{int} \ (A)$.

4) If $A \leq B$ then $\gamma - \text{int} \ (A) \leq \gamma - \text{int} \ (B)$.

5) $\gamma - \text{int} \ (A \wedge B) = \gamma \text{ int} \ (A) \wedge \text{ int} \ (B)$.

6) $\gamma \text{ int} \ (A \vee B) \geq \gamma \text{ int} \ (A) \vee \gamma \text{ int} \ (B)$.

3 Fuzzy $\gamma^*$-Semi open set

Definition 3.1. Let $(X, \tau)$ be a fuzzy topological space then a fuzzy subset $A$ of a fuzzy topological space $(X, \tau)$ is fuzzy $\gamma^*$ semi open set if $\text{int} \ (A) \leq \text{cl} \ (\gamma - \text{int} \ A)$.
Example 3.2. \( X = \{a, b, c\} \) and the topology 
\( \tau = \{0, 1, \{a, 2, b, 3, c\}, \{a, 4, b, 7, c\}, \{a, 2, b, 3, c\}, \{a, 4, b, 7, c, 5\}\) and \( \tau^c = \{0, 1, \{a, 8, b, 7, c\}, \{a, 6, b, 3, c, 7\}, \{a, 8, b, 7, c\}, \{a, 6, b, 3, c, 5\}\}. 
Let \( A = \{a, 4, b, 7, c\} \), \( \text{Int} (A) = \{a, 4, b, 7, c\} \) and \( \text{cl} (\gamma - \text{int} (A)) = \{a, 8, b, 7, c\} \). Hence \( A \) is fuzzy \( \gamma^* \) semi open set.

Proposition 3.3. Let \((X, \tau)\) be a fuzzy topological space then the union of any two fuzzy \( \gamma^* \) semi open sets is a fuzzy \( \gamma^* \) semi open set.

Proof. Let \( A_1, A_2 \) be two fuzzy \( \gamma^* \) semi open sets by the definition of \( \gamma^* \) semi open set i.e., \( \text{int} (A_1) \leq \text{cl} (\gamma \text{ int} (A_1)) \) and \( \text{int} (A_2) \leq \text{cl} (\gamma \text{ int} (A_2)) \). Therefore \( \text{int} (A_1) \lor \text{int} (A_2) \leq (\text{cl} (\gamma \text{ int} (A_1)) \lor (\text{cl} (\gamma \text{ int} (A_2))) \leq \text{cl} (\gamma - \text{int} (A_1) \lor \gamma - \text{int} (A_2)) \). Therefore \( \text{int} (A_1 \lor A_2) \leq \text{cl} (\gamma - \text{int} (A_1 \lor A_2)) \). 

Theorem 3.4. Let \((X, \tau)\) be a fuzzy topological space and let \( \{A_\alpha\}_{\alpha \in \Delta} \) be a collection of fuzzy \( \gamma^* \)-semi open sets in a fuzzy topological space \( X \), then \( \bigvee_{\alpha \in \Delta} A_\alpha \) is fuzzy \( \gamma^* \)-semi open.

Proof. Let \( \Delta \) be a collection of fuzzy \( \gamma^* \)-semi open sets of a fuzzy topological space \((X, \tau)\), then by using Definition 3.1, for each \( \alpha \in \Delta \), \( \text{int} (A_\alpha) \leq \text{cl} (\gamma - \text{int} (A_\alpha)) \).
Thus, \( \bigvee_{\alpha \in \Delta} A_\alpha \leq \bigvee_{\alpha \in \Delta} \text{cl} (\gamma - \text{int} (A_\alpha)) \). Since, \( \forall \text{cl} (A_\alpha) \leq \text{cl} (\forall A_\alpha) \).
i.e., \( \text{int} \left( \bigvee_{\alpha \in \Delta} A_\alpha \right) \leq \text{cl} \left( \bigvee_{\alpha \in \Delta} \gamma \text{ int} (A_\alpha) \right) \).
i.e., \( \text{int} \left( \bigvee_{\alpha \in \Delta} A_\alpha \right) \leq \text{cl} \left( \gamma \text{ int} \left( \bigvee_{\alpha \in \Delta} A_\alpha \right) \right) \). Thus the arbitrary union of fuzzy \( \gamma^* \)-semi open sets in fuzzy \( \gamma^* \)-semi open.

Theorem 3.5. Let \((X, \tau)\) and \((Y, \sigma)\) be any two fuzzy topological spaces such that \( X \) is product related to \( Y \) then the product \( A_1 \times A_2 \) of fuzzy \( \gamma \)-open set \( A_1 \) of \( X \) and fuzzy \( \gamma \)-open set \( A_2 \) of \( Y \) is fuzzy \( \gamma \)-open set of the fuzzy product space \( X \times Y \).

Proof. Let \( A_1 \) be a fuzzy \( \gamma \)-open subset of \( X \) and \( A_2 \) be a fuzzy \( \gamma \)-open subset of \( Y \), then by Definition 2.7 (4). we have 
\( A_1 \leq \text{int} (\text{cl} (A_1)) \lor \text{cl} (\text{int} (A_1)) \) and \( A_2 \leq \text{int} (\text{cl} (A_2)) \lor \text{cl} (\text{int} (A_2)) \).
Now, \( A_1 \times A_2 \leq \left( \text{int} (\text{cl} (A_1)) \lor \text{cl} (\text{int} (A_1)) \right) \times \left( \text{int} (\text{cl} (A_2)) \lor \text{cl} (\text{int} (A_2)) \right) \) then \( A_1 \times A_2 \leq \min \{ (\text{int} (\text{cl} (A_1)) \lor \text{cl} (\text{int} (A_1)) \}, (\text{int} (\text{cl} (A_2)) \lor \text{cl} (\text{int} (A_2))) \} \leq \text{int} (\text{cl} (A_1)) \lor \text{cl} (\text{int} (A_1)) \lor \text{int} (\text{cl} (A_2)) \lor \text{cl} (\text{int} (A_2)) \lor \text{cl} (\text{int} (A_1)) \lor \text{int} (\text{cl} (A_2)) \lor \text{cl} (\text{int} (A_1)) \lor \text{int} (\text{cl} (A_2)) \) = \( \text{int} (\text{cl} (A_1)) \lor \text{cl} (\text{int} (A_2)) \lor \text{int} (\text{cl} (A_1)) \lor \text{int} (\text{cl} (A_2)) \).
Therefore $A_1 \times A_2$ is fuzzy $\gamma$–open in the fuzzy product space $X \times Y$.

**Theorem 3.6.** Let $(X, \tau)$ and $(y, \sigma)$ be any two fuzzy topological spaces such that $X$ is product related to $Y$ then the product $A_1 \times A_2$ of a fuzzy $\gamma^*$–semi open set $A_1$ of $X$ and a fuzzy $\gamma^*$–semi open set $A_2$ of $Y$ is fuzzy $\gamma^*$– semi open set of the fuzzy product space $X \times Y$.

Proof. Let $A_1$ be a fuzzy $\gamma^*$–semi open subset of $X$ and $A_2$ be a fuzzy $\gamma^*$–semi open subset of $Y$, then by Definition 3.1 $\text{int} \ (A_1) \leq \text{cl} \ (\text{int} \ (A_1))$ and $\text{int} \ (A_2) \leq \text{cl} \ (\text{int} \ (A_2))$, i.e. $\text{int} \ (A_1) \times \text{int} \ (A_2) \leq \text{cl} \ (\text{int} \ (A_1)) \times \text{cl} \ (\text{int} \ (A_2))$.

By preceding Lemma 2.9 , $\text{int} \ (A_1 \times A_2) \leq \text{cl} \ (\text{int} \ (A_1) \times (\text{int} \ (A_1)))$.

Therefore $\text{int} \ (A_1 \times A_2) \leq \text{cl} \ (\text{int} \ (A_1 \times A_2))$. Hence $A_1 \times A_2$ is fuzzy $\gamma^*$–semi open in the fuzzy product space $X \times Y$.

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**4 Fuzzy $\gamma^*$– Semi Closed Sets**

**Definition 4.1.** A fuzzy subset $A$ of a fuzzy topological space $(X, \tau)$ is fuzzy $\gamma^*$–semi closed if $\text{cl} \ (A) \geq \text{int} \ (\gamma - \text{cl} \ (A))$.

**Example 4.2.** Let $X = \{a, b, c\}$ and the topology $\tau = \{0, 1, \{a, b, c\}, \{a, b, c\}, \{a, b, c\}, \{a, b, c\}\}$ and $\tau^* = \{0, 1, \{a, b, c\}, \{a, b, c\}, \{a, b, c\}, \{a, b, c\}\}$

Let $A = \{a_4, b_5, c_3\}, \gamma - \text{cl} \ (A) = \{a_8, b_9, c_8\}, \text{int} \ (\gamma \ - \text{cl} \ (A)) = \{a_2, b_7, c_5\}$. Therefore $A$ is $\gamma^*$–semi closed.

**Proposition 4.3.** Let $(X, \tau)$ be a fuzzy topological space, then the intersection of two fuzzy $\gamma^*$ semi closed sets is fuzzy $\gamma^*$ semi closed set in the fuzzy topological space $(X, \tau)$.

Proof. Let $A_1, A_2$ be two fuzzy $\gamma^*$ semi closed set. $\text{cl} \ (A_1) \geq \text{int} \ (\gamma - \text{cl} \ (A_1))$ and $\text{cl} \ (A_2) \geq \text{int} \ (\gamma - \text{cl} \ (A_2))$.

$\therefore \text{cl} \ (A_1) \cap \text{cl} \ (A_2) \geq \text{int} \ (\gamma - \text{cl} \ (A_1)) \cap \text{int} \ (\gamma - \text{cl} \ (A_2)) \geq \text{int} \ ((\gamma - \text{cl} \ (A_1)) \cap (\gamma - \text{cl} \ (A_2)))$.

Therefore $\text{cl} \ (A_1 \wedge A_2) \geq \text{int} \ (\gamma - \text{cl} \ (A_1 \wedge A_2))$. Hence $A_1 \wedge A_2$ is $\gamma^*$ semi closed set.

**Theorem 4.4.** Let $(X, \tau)$ be a fuzzy topological space and let $\{A_\alpha\}_{\alpha \in \Delta}$ be a collection of fuzzy $\gamma^*$–semi closed sets in a fuzzy
topological space $X$, then $\bigwedge_{\alpha \in \Delta} A_{\alpha}$ is fuzzy $\gamma^*$-semi closed for each $\alpha \in \Delta$.

Proof. Let $\Delta$ be a collection of fuzzy $\gamma^*$-semi closed sets of a fuzzy topological space $(X, \tau)$, then by using Definition 4.1, for each $\alpha \in \Delta$, $\text{cl}(A_{\alpha}) \geq \text{int}(\gamma - \text{cl}(A_{\alpha}))$.

Thus, $\text{cl}(\bigwedge_{\alpha \in \Delta} A_{\alpha}) \geq \bigwedge_{\alpha \in \Delta} (\text{int}(\gamma - \text{cl}(A_{\alpha})))$.

Since, $\bigwedge \text{int}(A_{\alpha}) \geq \text{int}(\bigwedge A_{\alpha})$.

i.e., $\text{cl}(\bigwedge_{\alpha \in \Delta} A_{\alpha}) \geq \bigwedge_{\alpha \in \Delta} (\text{int}(\gamma - \text{cl}(A_{\alpha})))$.

Therefore $\text{cl}(\bigwedge_{\alpha \in \Delta} A_{\alpha}) \geq \bigwedge (\text{int}(\gamma - \text{cl}(\bigwedge_{\alpha \in \Delta} A_{\alpha})))$.

Thus the arbitrary intersection of fuzzy $\gamma^*$-semi closed sets is fuzzy $\gamma^*$-semi closed.

\begin{theorem} \label{thm4.5} \textit{Let $(X, \tau)$ and $(y, \sigma)$ be any two fuzzy topological spaces such that $X$ is product related to $Y$ then the product $A_1 \times A_2$ of a fuzzy $\gamma^*$-semi closed set $A_1$ of $X$ and a fuzzy $\gamma^*$-semi closed set $A_2$ of $Y$ is fuzzy $\gamma^*$-semi closed set of the fuzzy product space $X \times Y$.} \end{theorem}

Proof. Let $A_1$ be a fuzzy $\gamma^*$-semi closed subset of $X$ and $A_2$ be a fuzzy $\gamma^*$-semi closed subset of $Y$, then by Definition 4.1 $\text{cl}(A_1) \geq \text{int}(\gamma \text{cl}(A_1))$ and $\text{cl}(A_2) \geq \text{int}(\gamma \text{cl}(A_2))$.

Therefore $\text{cl}(A_1) \times \text{cl}(A_2) \geq (\text{int}(\gamma \text{cl}(A_1))) \times (\text{int}(\gamma \text{cl}(A_2)))$.

By preceding Lemma 2.9, $\text{cl}(A_1 \times A_2) \geq \text{int}(\gamma \text{cl}(A_1 \times A_2)) = \text{cl}(A_1 \times A_2)$.

Therefore $\text{cl}(A_1 \times A_2) \geq \text{int}(\gamma \text{cl}(A_1 \times A_2))$ i.e $A_1 \times A_2$ is fuzzy $\gamma^*$-semi closed in the fuzzy product space $X \times Y$.

\section{Conclusion}

In general fuzzy topology Fuzzy $\gamma$-closed and $\gamma$-open sets are major role. Since its inception several weak forms of fuzzy $\gamma$-closed sets and $\gamma$-open sets have been introduced in general fuzzy topology. The present paper investigated in new weak forms Fuzzy $\gamma^*$-fuzzy semi open sets and Fuzzy $\gamma^*$-closed set in fuzzy topological spaces. Some basic properties and set product spaces are discussed. Many examples had been given to justify the results.
References


