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$L'(2,1)-$Edge Coloring of Trees and Cartesian Product of Path Graphs

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Abstract

Multi-level distance edge labeling of graph is introduced in this paper. We color the edges with nonnegative integers. If the edges are incident, then their color difference is at least 2 and if they are separated by exactly a single edge, then their colors must be distinct. The smallest among the largest colors of colorings is defined as the color span of this coloring. We find out in this paper the color span of certain trees and the cartesian product of paths.

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1 Introduction

A graph G consists of a finite nonempty set V of objects called *vertices* and a set E of 2-element subsets of V called *edges*.[1] If $e = uv$ is an edge of G, then u and v are *adjacent vertices*. Also, the vertex u and the edge e(as well as v and e)are said to be *incident* with each other. Distinct edges incident with a common vertex are *adjacent edges*. Readers may refer to [1] or [2] for the graph related terms used in this paper.

A graph G is *connected* if G contains a $u-v$ path for every pair u, v of distinct vertices of G.[1] The *distance* between u and v is the smallest length of among all the $u-v$ paths in G and is denoted by $d(u, v)$. The greatest distance between any two vertices of a connected graph G is called the *diameter* of G and is denoted by $diam(G)$.

For two graphs G and H, the *Cartesian product* $G \times H$ has vertex set $V(G \times H) = V(G) \times V(H)$, that is, every vertex of $G \times H$ is an ordered pair (u, v), where $u \in V(G)$ and $v \in V(H)$. Two distinct vertices (u, v) and (x, y) are adjacent in $G \times H$ if either $u = x$ and $vy \in E(H)$ or $v = y$ and $ux \in E(G)$.[1] A two-dimensional grid graph $G_{m,n}$ is the cartesian product $P_m \times P_n$ of the path graphs P_m and P_n.

As seen in [4] if G is a connected graph and $e_1 = (u_1, v_1), e_2 = (u_2, v_2)$ are two edges of G, then the *distance between edges or edge distance* of e_1 and e_2 is defined as

$$ed(e_1, e_2) = min\{d(u_1, u_2), d(u_1, v_2), d(v_1, u_2), d(v_1, v_2)\}.$$

If $ed(e_1, e_2) = 0$, then these edges are called *neighbor edges.*

Let c be a coloring of the vertices of a graph G such that for all vertices u and v of G, $|c(u) - c(w)| \geq s$ if $d(u, w) = 1$ and $|c(u) - c(w)| \geq t$ if $d(u, w) = 2$ for nonnegative integers s and t. This type of coloring is known as an $L(s, t)-$*coloring* of a graph G.[3]

In this paper, we study the $L^{'}(2, 1)$ edge coloring of a graph G. The $L^{'}(2, 1)$ edge coloring of a graph G is an assignment of non-negative integers to the edges e_1 and e_2 of G such that $|c(e_1) - c(e_2)| \geq 2$ if $ed(e_1, e_2) = 0$ and $|c(e_1) - c(e_2)| \geq 1$ if $ed(e_1, e_2) = 1$. No restriction is placed on colors assigned to edges at distance 2 or more. We now define the $L^{'}(2, 1)$ edge coloring number, $\lambda^{'}(G)$ for a graph.

2 The c-edge span of G

For an $L^{'}(2, 1)-$ edge coloring of G, the c-edge-span of G is the maximum value of $|c(e_1) - c(e_2)|$ over all pairs of edges e_1 and e_2 of $E(G)$. It is denoted by $\lambda^{'}_{2,1}(c)$.

The $L^{'}(2, 1)-$ edge coloring number denoted by $\lambda^{'}(G)$ is the smallest positive integer k such that there exists an $L^{'}(2, 1)-$ edge coloring $c : E(G) \to \{0, 1, 2, ..., k\}$.

We now see the $L^{'}(2, 1)-$ edge coloring number of path and stars.

Proposition 1. *Let P_k be the path graph on k vertices. Considering the $L^{'}(2, 1)-$ edge coloring of P_k, we see that $\lambda^{'}(P_3) = 2$, $\lambda^{'}(P_4) = \lambda^{'}(P_5) = 3$ and $\lambda^{'}(P_k) = 4$ for $k \geq 6$.*

Proof. Line graph of path is another path with the length reduced by one unit. Hence, the proof is similar to that of the $L(2, 1)$ vertex coloring of path graph. Hence, we omit the proof here. □

Theorem 2. *The edge coloring number of a star graph, $\lambda^{'}(K_{1,\Delta}) = 2(\Delta - 1)$.*

Proof. When $\Delta = 1$, we get P_2.

Hence, we onsider $\Delta \geq 2$. Then the edge coloring of $K_{1,\Delta}$ assigns colors $0, 2, 4, ..., (2\Delta - 2)$ to the edges of $K_{1,\Delta}$ satisfying the conditions of $L^{'}(2, 1)-$edge coloring. Hence, we observe that

$$\lambda^{'}(K_{1,\Delta}) \leq 2\Delta - 2 \qquad (1)$$

Suppose $\lambda^{'}(K_{1,\Delta}) \leq 2\Delta - 3 < 2\Delta - 2$. This implies there exists an $L^{'}(2, 1)$edge coloring of $K_{1,\Delta}$ using colors from the color set $0, 1, 2, ..., 2\Delta - 3$. As the edge distance between every edges in $K_{1,\Delta}$ is zero, the color distance between every pair of edges must be at least two. So if the first edge is colored i, the other edges must be colored $i+2$, $i+4$,....

So at most $(\frac{2\Delta-3}{2}+1)$ colors can be assigned to the edges. That is, the maximum number of colors that can be assigned to Δ edges is less than Δ. That is, at least one edge will have the same color, which is a contradiction as $diam(K_{1,\Delta})$=2. Hence, $\lambda^{'}(K_{1,\Delta})$ is not less than $2\Delta-2$. Therefore, $\lambda^{'}(K_{1,\Delta})=2(\Delta-1)$. □

3 Bounds of the $L^{'}(2,1)-$ Edge Coloring Number of trees with maximum degree Δ

Theorem 3. *For any tree T with maximum degree Δ, the $L^{'}(2,1)-$ edge coloring number is given by* $(2\Delta-2)\leq\lambda^{'}(T)\leq 2\Delta$.

Proof. As $L'(2,1)$ edge coloring depends on the distance between vertices of maximum degree in T, we categorize the trees based on the vertices of maximum degree and the distance between them. Let S_e be the set of even colors and S_o be the set of odd colors such that $|S_e|=2\Delta-2$ and $|S_o|=\Delta-1$.

Case 1. T has exactly one vertex of maximum degree Δ.

As $K_{1,\Delta}$ is a subgraph of T, we see that $\lambda^{'}(T)\geq 2\Delta-2$.

Let u be the vertex of maximum degree Δ in T. Color all the edges incident to u using the colors from S_e. The immediate edges which are not colored can be colored using the colors from S_o and then from S_e. this gives the required $L'(2,1)$ edge coloring of T. Hence,

$$\lambda'(T)=2\Delta-2 \tag{1}$$

Case 2. T has exactly two vertices of maximum degree Δ at distance one or at distance 2.

Let u and $v$be the vertices of maximum degree Δ in T at distance one. As $K_{1,\Delta}$ is a subgraph of T, we see that

$$\lambda^{'}(T)\geq 2\Delta-2$$

Let the Δ edges incident at u be colored using even colors from the set $S_e=\{0,2,4,...,2\Delta-2\}$.Clearly there exists $(\Delta-1)$ odd numbers from 0 to $2\Delta-2$. Also, at most Δ edges are incident to v or at most $(\Delta-1)$ edges are incident to v and not incident to u as Δ is the maximum degree in T. Let the above mentioned $(\Delta-1)$ edges incident to v be $vv_1,vv_2,...,vv_{\Delta-1}$.We see that none of the above edges can be colored using the colors from the set S_e. Hence the above $(\Delta-1)$ edges incident to v need to be colored using the colors from the set $S_o=\{1,3,...,(2\Delta-3)\}$ where $|S_o|=(\Delta-1)$. As $\lambda^{'}(T)$ is the smallest color used among any $L^{'}(2,1)-$ edge coloring of T, let us color the edge $e=uv$ using the smallest color from S_e, say, 0. Then none of the edges incident to v, say, $vv_1,vv_2,...,vv_{\Delta-1}$ can be colored 1. That is, these edges can neither be given the color 1 nor any of the colors in S_e. Hence, only at most $(\Delta-2)$ colors from S_o can be used to color the $(\Delta-1)$ edges. Advertently, none of these edges can have the same color implies that the smallest number greater than $(2\Delta-2)$ can be used to color one of the remaining edges incident to v. That is, at most $(\Delta-1)$ odd numbers are used to color the edges incident to v and not incident to u. all other edges which are not incident to u and v can be colored using any of the colors from S_e or S_o, according to the $L^{'}(2,1)-$ edge coloring criteria. Hence the smallest color used in $L^{'}(2,1)-$ edge coloring of T is equal to Δ number of even colors and $(\Delta-1)$ number of odd colors. That is,

$$\lambda^{'}(T) = 2\Delta - 1 \tag{2}$$

Case 3. T has exactly two vertices of maximum degree Δ at distance greater than 2 or T has more than two adjacent vertices of degree Δ.

Let u, v and w be the vertices of degree Δ at distance one and two repectively from u. From *case(2)* we see that $\lambda^{'}(T) \geq 2\Delta - 1$. None of the edges incident to w can be given the same color as that of the edges incident to v. Let x be the edge incident to both v and w. As w cannot be assigned colors $x-1$ and $x+1$, we see that at least one more color is required to color the edges incident to w. that is, at least $(2\Delta-1)+1$ colors are required. As $\lambda^{'}(T)$ is the smallest among those, we have,

$$\lambda^{'}(T) = 2\Delta - 1 + 1 = 2\Delta \tag{3}$$

From the above equations we see that, $(2\Delta - 2) \leq \lambda^{'}(T) \leq 2\Delta$. □

4 $L^{'}(2,1)-$ Edge Coloring Number of Grids

A grid is a graph obtained by the Cartesian product of path with paths. We find the $L^{'}(2,1)-$ edge coloring number of smaller grids. We also conjecture the $L^{'}(2,1)-$ edge coloring number of large grids.

Theorem 4. *The $L^{'}(2,1)-$ edge coloring number of $P_2 \times P_m$,*

$$\lambda^{'}(P_2 \times P_m) = \begin{cases} 4 & \text{if } m = 2 \\ 6 & \text{if } m = 3 \\ 7 & \text{if } m \geq 4 \end{cases}$$

Proof. Let the graph be $P_2 \times P_2$. As the diameter of the graph is 2, every pair of edges is within distance 1. Hence, all edges should receive distinct colors. Coloring with $\{0, 1, 2, 3\}$ alone is not possible because the degree of every edge is 2. The optimal coloring is given in Figure 1. Let the graph be $P_2 \times P_3$. There are seven edges in $P_2 \times P_3$. If all the seven edges receive distinct colors, then we have $\lambda^{'}(P_2 \times P_3) \geq 6$. We observe that all the five edges which are marked (See Figure 2) should have distinct colors as they are withing an edge-distance 1. The two edges unmarked can at most receive the same colors.

We consider various cases of the unmarked edges receiving one of the colors from $\{0, 1, 2, 3, 4, 5\}$.

If both the unmarked edges receive the color 0, then the central edge only can receive color 1. In that case, no other edge can receive color 2. If the central edge receives color 2, then no other edge can receive colors 1 and 3.

If both the unmarked edges receive the color 1, then the central edge only can receive colors 0 and 2. If the central edge receives color 0, then color 2 is to be

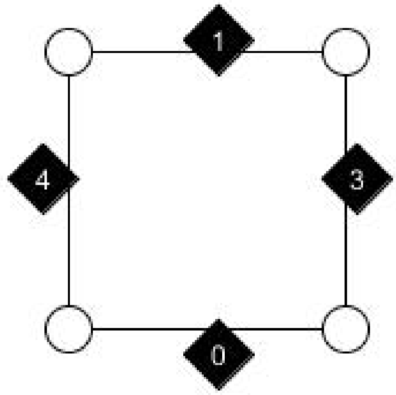

Figure 1: Edge coloring of $P_2 \times P_2$

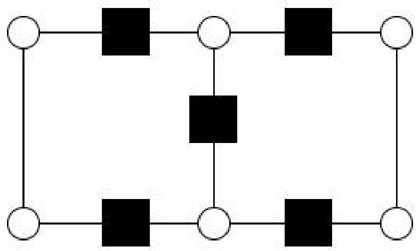

Figure 2: $P_2 \times P_3$

dropped to color the remaining edges. If the central edge receives color 2, then colors 0 and 3 are forbidden to color the remaining edges.

If both the unmarked edges receive the color 2, then the central edge only can receive colors 1 and 3. If the central edge receives color 1, then color 3 is to be dropped to color the remaining edges. If the central edge receives color 3, then colors 1 and 4 are forbidden to color the remaining edges.

If both the unmarked edges receive the color 3, then the central edge only can receive colors 2 and 4. If the central edge receives color 2, then color 4 is to be dropped to color the remaining edges. If the central edge receives color 4, then colors 2 and 5 are forbidden to color the remaining edges.

If both the unmarked edges receive the color 4, then the central edge only can receive colors 3 and 5. If the central edge receives color 3, then colors 2 and 5 are to be dropped to color the remaining edges. If the central edge receives color 5, then color 3 is forbidden to color the remaining edges.

If both the unmarked edges receive the color 5, then the central edge only can receive color 4. In that case, no other edge can receive color 3. If the central edge receives color 3, then no other edge can receive colors 2 and 4.

All the cases above imply that no two edges in the graph can receive the same colors from $\{0, 1, 2, 3, 4, 5\}$. Hence, as there are seven edges, $\lambda'(P_2 \times P_3) \geq 6$.

However, the edge coloring given in Figure 3 affirms that $\lambda'(P_2 \times P_3) = 6$.

Now, let us consider $P_2 \times P_4$ (See Figure 4).

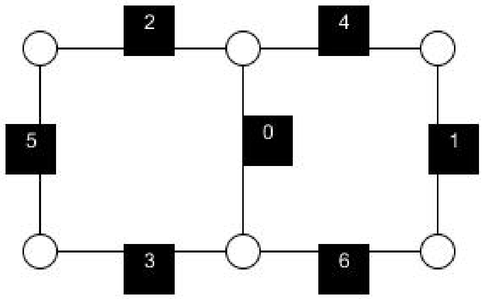

Figure 3: Edge coloring of $P_2 \times P_3$

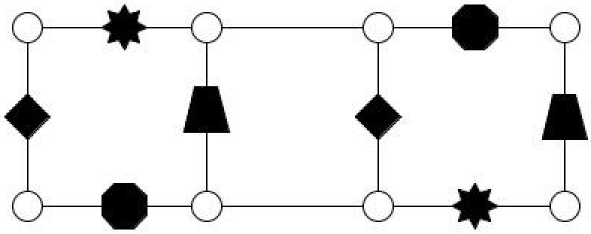

Figure 4: $P_2 \times P_4$

There are ten edges in $P_2 \times P_4$. Among them, there are four pairs of edges that are at distance 2 in the graph. Hence, in an optimal coloring, to each of these four pairs, we can assign similar colors. The pair of edges that can be given similar colors are marked with similar icons in Figure 4. Let the assigned colors to the four pairs of edges be $x-1$, x, $x+2$ and $x+3$.

There are two edges left with no colors assigned. None of these edges can receive any of the assigned colors or $x+1$ or $x+4$. Hence, the possible best colors are $x+5$ and $x+6$ (See Figure 5).

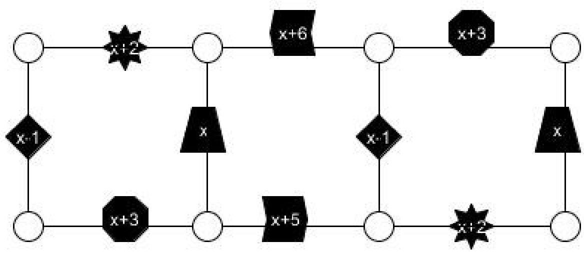

Figure 5: Edge coloring of $P_2 \times P_4$

Hence, $\lambda^{'}(P_2 \times P_4) = x+6-(x-1) = 7$.

In fact, for $P_2 \times P_m$ even when $m > 4$, the span is 7. The edge coloring is given in the following manner. We arrange the edges in to three sets. The vertical edges form the first set E_v. The top and bottom horizontal edges may be assigned to sets E_{ht} and E_{hb}, respectively. Assign all the edges of E_v the colors 0 and 1. Assign repeatedly, all the edges of E_{ht} and E_{hb} the colors {3,5,7,4,6} and {4,6,3,5,7}, respectively. This

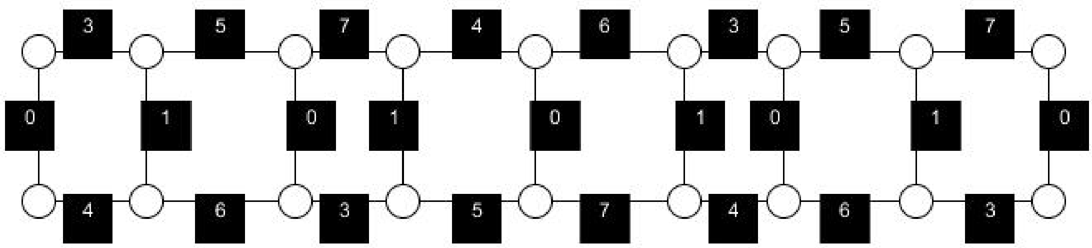

Figure 6: Edge coloring of $P_2 \times P_4$

gives an $L^{'}(2,1)$ edge coloring of the grid $P_2 \times P_m$. See Figure 6.

Hence, $\lambda^{'}(P_2 \times P_m) = 7$ if $m \geq 4$. □

We could color the edges of any grid with at most 11 colors. Hence, $\lambda^{'}$ for a grid is at most 10 (See Figure 7). Repeating this pattern in the style of a Latin Square, we can color the edges of a grid of any size. As a proper mathematical proof is still elusive, we conjecture the result as follows.

Conjecture 1. *For any rectangular grid $P_m \times P_n$,*

$$\lambda^{'}(P_m \times P_n) \leq 10 \quad \forall\ m,\ n.$$

5 Conclusion

In this paper we have introduced the edge-coloring version of the famous L(2,1)-vertex coloring of graphs. The $L^{'}(2,1)-$ Edge Coloring Number or $L^{'}(2,1)$-span of a graph is defined and is evaluated for some trees and the Cartesian product of path graphs. There is enough scope for doing much research in this field due to its importance in the channel assignment problems. Though in an exhaustive method, the proof of the conjecture is achievable, we feel that it is is worth attacking in a better robust mathematical way.

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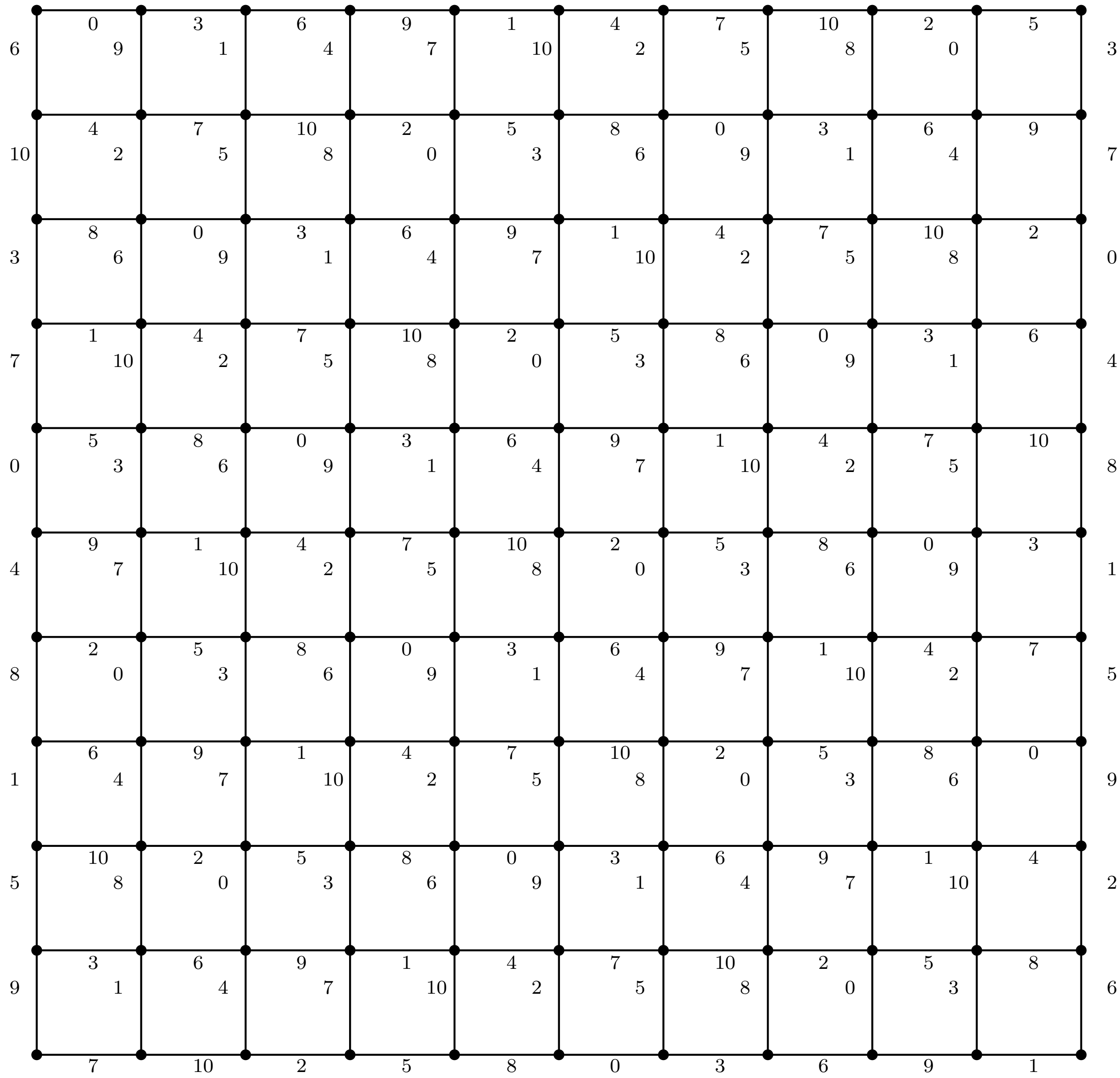

Figure 7: $P_{11} \times P_{11}$

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