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Diametral Paths in Total Graphs

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Abstract

The shortest path between two vertices which has length equal to diameter of that graph is a diametral path of that graph. The total graph $T(G)$ of a graph G is a graph that has vertex set $V(G)UE(G)$ and two vertices of $T(G)$ are adjacent whenever they are neighbors in G. There is an edge between two vertices in $T(G)$ if and only if there is edge-edge adjacency or edge-vertex incidence or vertex-vertex adjacency between the corresponding elements in G. In this paper, $diam(T(G))$ is proved as $diam(G)$ or $diam(G) + 1$.

AMS Subject Classification: 05C12

Key Words and Phrases: Diameter of a graph, peripheral vertices, Diametral path, Total graph

1 Introduction

The study on peripheral vertices and diametral paths gives us vital information on the structure of a graph and helps us address problems in transportation, distribution, designing, communication, team formation and event management. In analyzing and designing of networks, the concept of a diametral path which represents a critical route is of great significance. The concepts of paths, diameter, diametral paths, and their applications are investigated extensively in the literature [1], [2], [3], [6], [7] and [8]. Diametral path graphs are introduced and discussed in detail in [4]. The diameter gives critical information on how remotely placed the vertices are among themselves in a graph. The diametral path represents this critical path between such vertices. In transportation networks, this can be regarded as the unavoidable distance that needs to be covered to transport goods between such nodes. When the critical routes are known, effective decisions can be taken to improve the connectivity.

In this paper, a study on diameter related concepts are undertaken. In Section 3, the number of edges in the total graph of a connected graph is determined in terms of degree of its vertices. In Section 4, it is proved that for any connected graph G, $diam(T(G))$ is equal to $diam(G)$ or $diam(G) + 1$.

2 Terminologies

The definitions and results that are not elaborated in deatils are in the sense of [2] and [5]. The degree $d(v)$ of a vertex v of a graph G is the number of edges incident on v.[5] The length of a path is the number of edges on the path.[8] The distance $d_G(u, v)$ or $d(u, v)$ between two vertices $u, v \in V(G)$ in a graph G is the length of a shortest path between them.[8] The eccentricity of a vertex is the maximum of distances from it to all the other vertices of that graph.[5] While the diameter $diam(G)$ of a graph G is the maximum of the eccentricities of all vertices of that graph, the radius is the minimum of these. Peripheral vertices are the vertices of maximum eccentricity and central vertices are of minimum eccentricity. is Among the shortest path between two vertices which has the length equal to the diameter of that graph is known as a diametral path of that graph.[5]

The vertices and the edges of a graph are called its elements. Two elements of a graph are neighbors if they are either incident or adjacent.[4] The total graph $T(G)$ as introduced by F. Harary [5] has vertex set $V(G)UE(G)$ and two vertices of $T(G)$ are adjacent whenever they are neighbors in G. There is an edge between two vertices in $T(G)$ if and only if there is edge-edge adjacency or edge-vertex incidence or vertex-vertex adjacency between the corresponding elements in G. Hence the number of vertices in $T(G)$ is $n + m$ where n are the number of vertices of the original graph $G(n, m)$ and m are the new vertices representing the edges of the original graph. The degree sequence in total graph and their related cliques are investigated by Thomas and Varghese [9].

3 Total Graph

It is not difficult to observe that a graph is connected if and only if its total graph is connected. Hence, in the subsequent discussions, we are concerned only about connected graphs. Moreover, we do deal with only simple undirected graphs. The structure of the total graph can be analyzed by the following results on degrees of vertices and the number of edges in total graphs.

Lemma 1. *Let $u \in V(G)$ and $e = \{v, w\} \in E(G)$. Then for $u, e \in V(T(G))$, (1) $d_{T(G)}(u) = 2d_G(u)$ and (2) $d_{T(G)}(e) = d_G(v) + d_G(w)$.*

Proof. (1) Consider vertex u in $T(G)$ where $u \in V(G)$. In $T(G)$, due to vertex-vertex adjacency in G, u is adjacent to $d_G(u)$ vertices. Also due to edge-vertex incidence in G, u is adjacent to $d_G(u)$ vertices in $T(G)$. Hence the degree of each vertex u in $T(G)$ is $2(d_G(u))$.

(2) Consider vertex e in $T(G)$ where $e \in E(G)$. Due to edge-edge adjacency and edge-vertex incidence in G, vertex e is adjacent to $d_G(v) + d_G(w)$ vertices in $T(G)$ where $e \in E(G)$ and v, w are its endvertices in G. Hence degree of each vertex e in $T(G)$ is $d_G(v) + d_G(w)$.

□

Lemma 2. *The number of edges in the total graph $T(G)$ of a connected graph*

G is

$$\frac{1}{2}\sum_{v\in V(G)} d(v)(d(v)+2)$$

.

Proof. Consider graph G of order n and size m. The edges in $T(G)$ can be categorized in the following manner. Considering the vertex-vertex adjacency in G, there are m edges of G in $T(G)$. Since each edge of G is incident on its endvertices, there are $2m$ edges in $T(G)$ due to the edge-vertex incidence in G. Finally, since degree $d(v)$ of each vertex v in G is the number of edges incident on it, it gives us the number of edges adjacent to each other at each vertex in G. Also every two edges incident on a common vertex in G contribute to an edge in $T(G)$. Considering edge-edge adjacency in G, we get

$$\sum_{vV(G)} \binom{d(v)}{2}$$

edges in $T(G)$.

Hence the total number of edges is

$$\begin{aligned}
&= m + 2m + \sum_{v\in V(G)} \binom{d(v)}{2} \\
i.e., &= 3m + \sum_{v\in V(G)} \binom{d(v)}{2} \\
i.e., &= \frac{3}{2}\sum_{v\in V(G)} d(v) + \sum_{v\in V(G)} \frac{d(v)(d(v)-1)}{2} \\
i.e., &= \frac{1}{2}\sum_{v\in V(G)} 3d(v) + d(v)(d(v)-1) \\
i.e., &= \frac{1}{2}\sum_{v\in V(G)} d(v)(d(v)+2)
\end{aligned}$$

□

4 Diametral Paths in Total Graphs

In this section, for a graph and its total graph, the parameter diameter is analyzed.

Lemma 3. *For a graph G, $diam(G) \leq diam(T(G))$.*

Proof. For any $u, v \in V(G)$, $d_G(u, v) = d_{T(G)}(u, v)$. In other words, the distance between vertices of G remains the same in G and $T(G)$. Every pair of nonadjacent vertices in G continues to be nonadjacent in $T(G)$. Also, every diametral path of G is an induced path in G and continues to be an induced path in $T(G)$. Hence, $diam(G) \leq diam(T(G))$. □

It is a fact that in an odd cycle, from every vertex there are two distinct diametral paths to the endvertices of an edge. We can foresee a variation in diameter in the odd cycle and its total graph. This situation turns more complex if there are diametral paths from multiple vertices to the endvertices of an edge in a graph. The question that arises from this situation is about how much a graph and its total graph differ in diameter. In the following lemma, this variation of diameter in a graph and its total graph is analyzed in detail.

Lemma 4. *If a graph G has diametral paths from any vertex to the endvertices of any edge in G, then $diam(T(G)) = diam(G) + 1$.*

Proof. Consider the graph G. Let $diam(G) = D$. We prove the result by analysing various cases.

Case 1: *There are two diametral paths from a vertex to the endvertices of an edge.* Consider a vertex u and an edge e in G. Let the endvertices of e be a and b. Let the distinct diametral paths from the vertex u to the vertices a and b in G be $P_1 = \{u, u_1, u_2 \dots u_{D-1}, u_D = a\}$ and $P_2 = \{u, u'_1, u'_2, \dots u'_{D-1}, u'_D = b\}$, respectively. Due to edge-vertex incidence of G in $T(G)$, the vertex e is adjacent to a and b in $T(G)$. However, lemma 3 assures that the distances $d(u, a)$ and $d(u, b)$ are invariant in $T(G)$.

Hence $d(u, a) = d(u, b) = D$ and

$$d(u, e) = d(u, a) + d(a, e) = d(u, b) + d(b, e) = D + 1.$$

Hence we have $diam(T(G)) = D + 1$.

Case 2: *There are two non-adjacent vertices from each of which, there are two diametral paths each to the endvertices of a single edge.*

Let u and v be the two non-adjacent vertices in G. From each of them, there are two diametral paths each to the endvertices of an edge e. Obviously, $d(u, v)$ in $T(G)$ does not exceed the the diameter D of the graph G. Hence, using case 1, we conclude that $diam(T(G)) = D + 1$.

Case 3: *There are two adjacent vertices from each of which, there are two diametral paths each to the endvertices of a single edge.* Let $e' = uv$ in G.

Hence, in $T(G)$,

$$d(a, e') = d(a, u) + d(u, e') = d(a, v) + d(v, e') = D + 1.$$

.

Also in $T(G)$,

$$d(b, e') = d(b, u) + d(u, e') = d(b, v) + d(v, e') = D + 1.$$

Also the distance between e and e', $d(e, e')$ in $T(G)$ appears to be be $D+1+1 = D + 2$. But we prove that this is not true and $diam(T(G))$ is only $D + 1$.

From case 1, we have $P_1 = \{u, u_1, u_2 \ldots u_{D-1}, u_D = a\}$ and $P_2 = \{u, u'_1, u'_2, \ldots u'_{D-1}, u'_D = b\}$. In $T(G)$, by lemma 3 $d(u, a)$ is invariant. As per the construction of total graph, $P_1 = \{u, u_1, u_2 \ldots u_{D-1}, u_D = a\}$ induces D vertices $uu_1, u_1u_2 \ldots u_{D-2}u_{D-1}$ and $u_{D-1}u_D$.

Obviously they form a path of length $D - 1$. i.e., $d(uu_1, u_{D-1}u_D) = D - 1$. However in G, e', uu_1 and uu'_1 are adjacent edges. Hence, as vertices, e', uu_1 and uu'_1 are adjacent to each other and u in $T(G)$.

Therefore, $d(e', u_{D-1}u_D) = d(e', uu_1) + d(uu_1, u_{D-1}u_D) = 1 + D - 1 = D$.

This implies that $d(e', e) = d(e', u_{D-1}u_D) + d(u_{D-1}u_D, e) = D + 1$. This is because $u_{D-1}u_D$ and $u_Db = ab$ are adjacent edges in G and are adjacent as vertices in $T(G)$.

Case 4: Suppose in G, there exists two other vertices s and t such that there are distinct diametral paths from u to s, u to t, v to s, v to t, s to a, s to b, t to a and t to b. From the above cases, we can conclude that the $diam(T(G))$ is $D + 1$. Hence, the proof.

□

We now have the major result of this paper.

Theorem 1. *For a graph G, $diam(T(G))$ is equal to $diam(G)$ or $diam(G) + 1$.*

Proof. Consider a $G(n, m)$ with vertices $v_1, v_2 \ldots v_n$ and edges $e_1, e_2 \ldots e_m$. Due to vertex-vertex adjacency of G in $T(G)$, the distance between $v_i's$ in $T(G)$ remains unaffected. Due to edge-edge adjacency of G in $T(G)$, the distance between e_i's in $T(G)$ remains unaffected. If there is a vertex v with diametral paths to the endvertices of an edge e in G, then $diam(T(G)) = diam(G) + 1$. (**Lemma** 4) Otherwise, diameter remains unaffected.
Hence $diam(T(G))$ is equal to $diam(G)$ or $diam(G) + 1$. □

We see two illustrations of this theorem now.

Example 1. *Consider a graph G of order five and size five as seen in Figure 1. It can be noted that $diam(G(5, 5)) = 2$. It has four diametral paths $\{u, b, v\}, \{u, b, w\}, \{a, b, v\}$ and $\{a, b, w\}$. There are two diametral paths from u to endvertices v and w of an edge e_1. Hence it can be noted that $diam(T(G(5, 5))) = 2 + 1 = 3$ as seen in Figure 2.*

5 Conclusion

In this paper, the nature of diametral paths is determined for certain classes of graphs and their total graphs. The focus of further research would be to find the decomposition of total graphs of these classes into diametral paths.

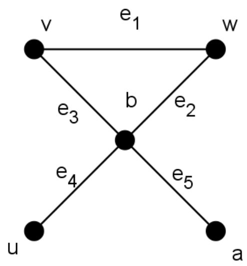

Figure 1: $G(5,5)$

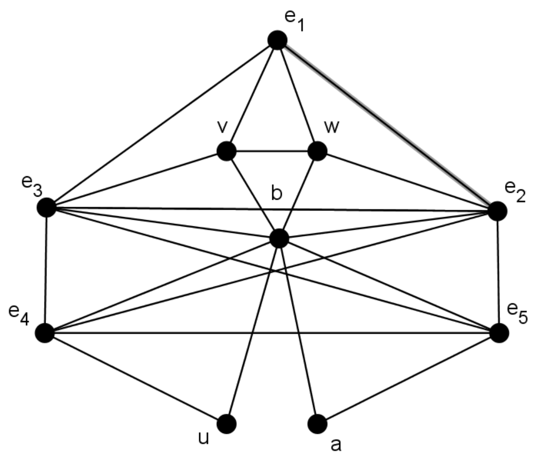

Figure 2: Path $T(G(5,5))$

References

[1] J. C. Bermond and J. Bond and M. Paoli and C. Peyrat, *Graphs and Interconnection Networks: Diameter and vulnerability*, Cambridge University Press, Cambridge (2003).

[2] F. Buckley and F. Harary, *Distance in Graphs*, Perseus Books (1990).

[3] F. Buckley and M. Lewinter, Graphs with all diametral paths through distant central vertices, *Mathematical and Computer Modelling*, Elsevier Inc. **17**, (1993), 35-41.

[4] J. S. Deogun and D. Kratsch, Diametral Path Graphs, *Graph-Theoretic Concepts in Computer Science*, **1017**, (1995), 344-357.

[5] F. Harary, *Graph Theory*, Narosa Publishing House, New Delhi (2001).

[6] T. A. Mangam and J. V. Kureethara, Diametral Paths in Total Graphs of Complete Graphs, Complete Bipartite Graphs and Wheels, *Int. J. Civil Eng. Tech.*, **8** (5), (2017), 1212-1219.

[7] T. A. Mangam and L. S. Reddy, Weighted peripheral graph, *Indian J. Sc. Tech.*, **9** (6), (2016), 1-5.

[8] O. Ore, Diameters in Graphs, *J. Combinatorial Theory*, **5**, (1968), 75-81.

[9] J. Thomas and J. Varghese, Decomposition in Total Graphs, *Advanced Modelling and Optimization*, **15** (1), (2013), 81-84.