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Decomposition of Hypercubes

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Abstract

A hypercube Q_{2n} where n is a positive integer can be decomposed into 4^{n-1} copies of n-fan $F_{n,4}$ and a hypercube Q_{2n+1} can be decomposed into maximum 4^{n-1} copies of double n-fan $F^*_{n,4}$ and 3.4^{n-1} copies of K_2.
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1 Introduction

The study on n-dimensional hypercube Q_n is made extensively.Hypercubes are fundamental structures used in communication as well as in coding theory. The n dimensional cube or *hypercube* Q_n is the simple graph whose vertices are the n-tuples with entries in $\{0, 1\}$ and whose edges are the pairs of n-tuples that differ in exactly one position. Fink and Ramras independently proved that Q_n can be edge decomposed into 2^{n-1} isomorphic copies of any tree on n edges [2] [4]. Wagner and Wild showed that Q_n is edge decomposable into n copies of a specific tree on 2^{n-1} edges [5]. Mollard and Ramras found edge decomposition of Q_n into copies of P_4, the path on 4 edges, for all n $\geq$ 5 [3]. Some other researchers, Anick and Ramras have worked on edge decomposition of hypercubes into paths of equal length [1].

A *decomposition* of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list. An *n-fan* $F_{n,a}$ is a graph with n cycles of length a attached to a common vertex called the root vertex.

A *double n-fan* $F^*_{n,a}$ is a graph constructed by adding an edge between the two copies of an n-fan graph $F_{n,a}$ at its root vertex.

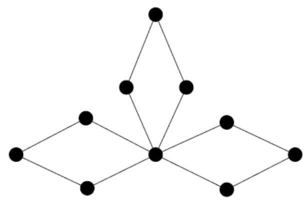

Figure 1: $F_{3,4}$

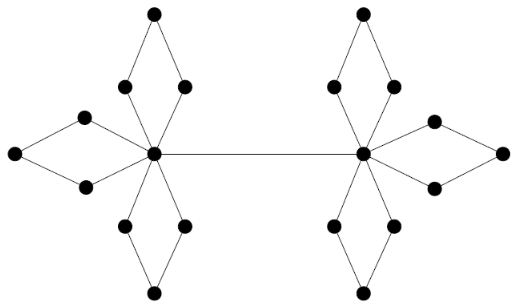

Figure 2: $F^*_{3,4}$

2 Main Theorems

Theorem 1. *The Hypercube Q_{2n} can be decomposed into 4^{n-1} copies of $F_{n,4}$.*

Proof. The hypercube Q_{2n} has $n.2^{2n}$ edges. Each copy of an n-fan $F_{n,4}$ has $4n$ edges, since we have 4^{n-1} copies of $F_{n,4}$ the total number of edges would be $4^{n-1}.4n$ which is the total edges in a hypercube Q_{2n}.

In the case of a hypercube Q_{2n} where n is a positive integer, each vertex is a *2n*- tuple given by $(v_1v_2...v_n...v_{2n})$ where each $v_i \in \{0,1\}$. Moreover if $v_i = 0$ then $\overline{v_i} = 1$. Let the adjacent bits of the array of *2n* bits be paired to form n blocks represented as $(b_1b_2...b_n)$ where each block $b_j = v_{2j-1}v_{2j}$ for $j \in \{1,2,...n\}$. To have the decomposition we need to identify

1. 4^{n-1} root vertices and
2. the n-cycles of length 4 attached to each root vertex.

The labels of the root vertices are generated in a table having n columns corresponding to n blocks and 4^{n-1} rows. The entry in one complete row will give the label of each root vertex. Let $b_{i,j}$ denote the entry corresponding to the i^{th} block and the j^{th} row in the table (Table 1). The entry $b_{i,j}$ is operated with an operator x denoted as $b^x_{i,j}$ where x takes vales 1, 2, 3 and 4. The action of the operator x is as follows

$$\left.\begin{aligned} b^1_{i,j} &= v_{2i-1}\ \ v_{2i} \\ b^2_{i,j} &= \overline{v_{2i-1}}\ \ v_{2i} \\ b^3_{i,j} &= \overline{v_{2i-1}}\ \ \overline{v_{2i}} \\ b^4_{i,j} &= v_{2i-1}\ \ \overline{v_{2i}} \end{aligned}\right\} \tag{1}$$

where $i \in \{1,2,...n\}$ and $j \in \{1,2,...4^{n-1}\}$

Generation of 4^{n-1} root vertices

Choose any vertex from the hypercube Q_{2n} say $(b_1\ b_2\ ...\ b_n)$ where each b_i is a block of two bits. Without laws of generality we can relabel this vertex as $b^1_{1,1}\ b^1_{2,1}$ ... $b^1_{n,1}$. Obviously this is the first entry in the table (Table 1) corresponding to the first row. Note that the valve of the operator x is 1 in the first row.

In the column corresponding to block b_1 as we have 4^{n-1} rows and the operator has four possible values we partition the rows into 4 parts namely, 1 upto 4^{n-2}, $4^{n-2}+1$ upto 2.4^{n-2}, $2.4^{n-2}+1$ upto 3.4^{n-2}, $3.4^{n-2}+1$ upto $4.4^{n-2} = 4^{n-1}$. In general for a block b_i corresponding to the i^{th} column for $i \in \{1, 2, ..., n-1\}$ we partition the total rows into 4^i equal parts. Each part is considered to be an interval. In the case of any block b_i in the table the operator x changes from $1 \to 2 \to 3 \to 4 \to 1$ an so on. The operator x changes after each interval and continues to be the same in the current interval. This process terminates once all the rows are filled.
To fill the entries $b^x_{n,j}$ corresponding to the n^{th} column for $j \in \{1, 2, ...4^{n-1}\}$ we find an a such that if $j \equiv a \pmod{16}$ and

- $a \in \{1, 8, 11, 14\}$ then $x = 1$.
- $a \in \{2, 5, 12, 15\}$ then $x = 2$.
- $a \in \{3, 6, 9, 0\}$ then $x = 3$.
- $a \in \{4, 7, 10, 13\}$ then $x = 4$ where x is the operator.

However, we know that Q_{2n} is the cartesian product of n copies of C_4, that is, $Q_{2n} = C_4 \Box C_4 \Box .... \Box C_4$. This implies that each induced cycle, i.e., C_4 in the cartesian product corresponds to a block b_i of Q_{2n}. Every cycle C_4 has four possible labels. In the above mentioned method we observe that at each stage keeping the block b_i fixed we can use all the possible four labels for the block b_{i+1} for $i \in \{1, 2, ..., n-2\}$. Thus entries in the block b_n is made in such a way no two root vertices generated in the table could be adjacent. This way we get all the 4^{n-1} root vertices.

Generation of the n C_4s attached to a given root vertex.

Step 1 Let $(b_1\ b_2\ ...\ b_n)$ be a root vertex of a hypercube Q_{2n} . Let C_4^k represents a cycle of length four which is the k^{th} leaf attached to the root vertex where $k \in \{1, 2, ...n\}$. The vertices belonging to the cycle in the k^{th} leaf can be generated by varying the block $b_k \in (b_1\ b_2\ ...\ b_n)$ while keeping all the other *n-1* blocks fixed. The block b_k is operated with the operator x as defined in equation (1). The cycle thus obtained is given below (figure 3)

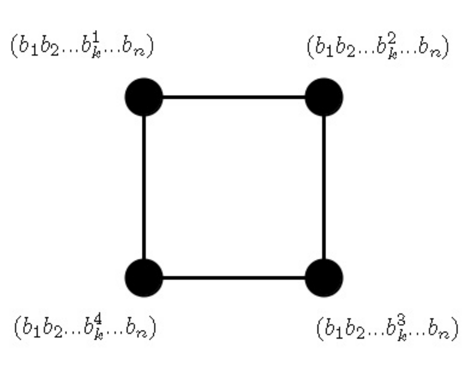

Figure 3: Generation of C_4^k

Step 2 Repeat the process for all the values of $k \in \{1, 2, ..., n\}$ to generate all the n leaves for a single root vertex.

Step 3 Repeat step 1 and 2 with all other root vertices to get 4^{n-1} copies of $F_{n,4}$. □

	b_1	b_2	b_3	.	.	b_{n-3}	b_{n-2}	b_{n-1}	b_n
1	$b^1_{1,1}$	$b^1_{2,1}$	$b^1_{3,1}$	.	.	$b^1_{n-3,1}$	$b^1_{n-2,1}$	$b^1_{n-1,1}$	$b^1_{n,1}$
2	$b^1_{1,2}$	.	.	.	.	$b^1_{n-3,2}$	$b^1_{n-2,2}$	$b^2_{n-1,2}$	$b^2_{n,2}$
3	$b^1_{1,3}$	.	.	.	.	$b^1_{n-3,3}$	$b^1_{n-2,3}$	$b^3_{n-1,3}$	$b^3_{n,3}$
4	$b^1_{1,4}$	.	.	.	.	$b^1_{n-3,4}$	$b^1_{n-2,4}$	$b^4_{n-1,4}$	$b^4_{n,4}$
5	$b^1_{1,5}$	.	.	.	.	$b^1_{n-3,5}$	$b^2_{n-2,5}$	$b^1_{n-1,5}$	$b^2_{n,5}$
6	$b^1_{1,6}$					$b^1_{n-3,6}$	$b^2_{n-2,6}$	$b^2_{n-1,6}$	$b^3_{n,6}$
7	$b^1_{1,7}$					$b^1_{n-3,7}$	$b^2_{n-2,7}$	$b^3_{n-1,7}$	$b^4_{n,7}$
8	$b^1_{1,8}$					$b^1_{n-3,8}$	$b^2_{n-2,8}$	$b^4_{n-1,8}$	$b^1_{n,8}$
9	$b^1_{1,9}$					$b^1_{n-3,9}$	$b^3_{n-2,9}$	$b^1_{n-1,9}$	$b^3_{n,9}$
10	.					$b^1_{n-3,10}$	$b^3_{n-2,10}$	$b^2_{n-1,10}$	$b^4_{n,10}$
11	.					$b^1_{n-3,11}$	$b^3_{n-2,11}$	$b^3_{n-1,11}$	$b^1_{n,11}$
12	.					$b^1_{n-3,12}$	$b^3_{n-2,12}$	$b^4_{n-1,12}$	$b^2_{n,12}$
13	.					$b^1_{n-3,13}$	$b^4_{n-2,13}$	$b^1_{n-1,13}$	$b^4_{n,13}$
14	.					$b^1_{n-3,14}$	$b^4_{n-2,14}$	$b^2_{n-1,14}$	$b^1_{n,14}$
15	.					$b^1_{n-3,15}$	$b^4_{n-2,15}$	$b^3_{n-1,15}$	$b^2_{n,15}$
16	.					$b^1_{n-3,16}$	$b^4_{n-2,16}$	$b^4_{n-1,16}$	$b^3_{n,16}$
.	.						.	.	.
.	.						.	.	.
.	.						.	.	.
4^{n-2}	$b^1_{1,4^{n-2}}$						.	.	.
.	.						.	.	.
.	.						.	.	.
.	.						.	.	.
2.4^{n-2}	$b^2_{1,2.4^{n-2}}$						.	.	.
.	.						.	.	.
.	.						.	.	.
.	.						.	.	.
3.4^{n-2}	$b^3_{1,3.4^{n-2}}$						.	.	.
.	.						.	.	.
.	.						.	.	.
.	.						.	.	.
4^{n-1}	$b^4_{1,4^{n-1}}$	$b^4_{2,4^{n-1}}$	.	.	.	.	$b^4_{n-2,4^{n-1}}$	$b^4_{n-1,4^{n-1}}$	$b^4_{n,4^{n-1}}$

Table 1: Generation of 4^{n-1} root vertices

Theorem 2. *The Hypercube Q_{2n+1} can be decomposed into 4^{n-1} copies of $F^*_{n,4}$ and 3.4^{n-1} copies of K_2.*

Proof. The hypercube Q_{2n+1} has $(2n+1).4^n$ edges. Each copy of a double n-fan $F^*_{n,4}$ has $8n+1$ edges. For 4^{n-1} copies of $F^*_{n,4}$ and 3.4^{n-1} copies of K_2 the total number of edges is $(4^{n-1})(8n+1) + 3.4^{n-1} = (2n+1).4^n$. This is the total edges in a hypercube Q_{2n+1}.

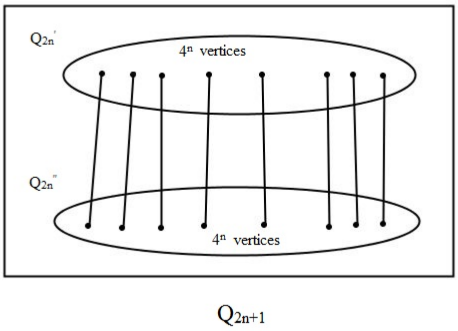

Figure 4: Q_{2n+1} is the cartesian product of Q_{2n} and K_2

The hypercube Q_{2n+1} is the cartesian product of Q_{2n} and K_2. Hence every Q_{2n+1} would have two copies of Q_{2n} and the edges between them. Let the two copies of Q_{2n} in Q_{2n+1} be Q'_{2n} and Q''_{2n}. Each vertex of a hypercube Q_{2n+1} is a 2n+1 tuple. Let the vertices of Q'_{2n} and Q''_{2n} in Q_{2n+1} be labeled as $(v_1v_2...v_n...v_{2n}\ v_{2n+1})$ and $(v_1v_2...v_n...v_{2n}\ \overline{v_{2n+1}})$ respectively. These vertex labels for Q'_{2n} and Q''_{2n} can be rewritten as $(b_1b_2...b_n\ v_{2n+1})$ and $(b_1b_2...b_n\ \overline{v_{2n+1}})$ where each block $b_j = v_{2j-1}v_{2j}$ for $j \in \{1, 2, ...n\}$.

From the definition of a double n-fan we note that each copy requires two root vertices. These root vertices are carefully identified in such a way that one root vertex is chosen from the copy of Q'_{2n} and the other one is chosen from the copy of Q''_{2n} .

Consider the blocks $(b_1b_2...b_n)$ from the vertex label of Q_{2n+1}. We can generate the labels of 4^{n-1} vertices which serve as root vertices for $F_{n,4}$ by theorem (1). Once these vertices are generated, then we add a bit v_{2n+1} for all the 4^{n-1} labels and set it to 0. This would give the required root vertices in Q'_{2n}. If the v^{th}_{2n+1} bit is set to 1 for all the 4^{n-1} vertex labels generated, then this would give the root vertices in Q''_{2n}.

For the above labels of the root vertices generated there will be 2.4^{n-1} copies of $F_{n,4}$ in Q_{2n+1} by theorem (1). From the labeling pattern of the root vertices in Q'_{2n} and Q''_{2n} we find there are 4^{n-1} copies of $F^*_{n,4}$. This is because there are 4^{n-1} pairs of root vertices which can be identified in such a way that in each pair, one root vertex is chosen from the copy Q'_{2n} and the other is chosen from the copy Q''_{2n}. Theses root vertices are adjacent since their vertex labels differ exactly by one bit.

By the construction of Q_{2n+1} we know that Q'_{2n} would have exactly 4^n vertices adjacent to the same number of vertices of Q''_{2n}. Since 4^{n-1} edges are already a part of 4^{n-1} copies of $F^*_{n,4}$ the remaining edges which are not a part of double n-fan decomposition is 3.4^{n-1}. Obviously this forms 3.4^{n-1} copies of K_2. Thus we get 4^{n-1} copies of double n-fan $F^*_{n,4}$ and 3.4^{n-1} copies of K_2.

The maximum number of copies of $F^*_{n,4}$ that can be obtained by the disjoint edge decomposition of a hypercube Q_{2n+1} is 4^{n-1}. To prove the above statement let us assume that, there exist atleast $4^{n-1}+1$ copies obtained by the decomposition of Q_{2n+1}.

$F^*_{n,4}$ cannot be a subgraph of Q'_{2n} or Q''_{2n}, that is, no copy of $F^*_{n,4}$ will lie completely in either Q'_{2n} or Q''_{2n}. Since there exist no two root vertices whose degrees are 2n+1 and which are adjacent to each other. The only possibility is to have one of the root vertex of $F^*_{n,4}$ in Q'_{2n} and the other in Q''_{2n}.

We assumed that the graph Q_{2n+1} has $4^{n-1}+1$ copies of $F^*_{n,4}$. Therefore each copy of $F^*_{n,4}$ would have its one root vertex in Q'_{2n} and the other one in Q''_{2n}. Thus there exist $4^{n-1}+1$ root vertices in Q'_{2n} as well as in Q''_{2n}. However the maximum number of non adjacent root vertices that can be there in the copy of Q'_{2n} or Q''_{2n} is 4^{n-1} by theorem (1). This shows that there is atleast a pair of root vertex which are adjacent in the copy of Q'_{2n} as well as in Q''_{2n}. Obviously due to the adjacent pair of root vertices, there exist less than $4^{n-1}+1$ copies of $F^*_{n,4}$ which is a contradiction to our assumption. Hence the maximum number of copies of $F^*_{n,4}$ that can be obtained by the edge decomposition of a hypercube Q_{2n+1} is 4^{n-1}. □

3 Conclusion

In this paper, we have obtained a decomposition of hypercube Q_{2n} into 4^{n-1} copies of n-fan $F_{n,4}$. We also have shown that a hypercube Q_{2n+1} can be decomposed into 4^{n-1} copies of double n-fan $F^*_{n,4}$ and 3.4^{n-1} copies of K_2.

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