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Convexity of Power Type Heron Mean With Respect to Power Mean

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Abstract

In this paper, the convexity and concavity of power type Heron mean with respect to Power mean and the results are interpreted with Vander monde's determinant. **AMS Subject Classification:** 26D10, 26D15.

Key Words and Phrases: Convexity, Concavity, Power mean, Vander Monde's determinant, Heron mean

1 Introduction

The well-known means are presented by Pappus of Alexandria in his books in the fourth century A.D., which is the main contribution of the ancient Greeks. In Pythagorean School, on the basis of proportion, ten Greek means are defined out of which six means are named and four means are un-named. Let us recall some of the popular mathematical means are Arithmetic mean$=A(a,b) = \frac{a+b}{2}$, Geometric mean$=G(a,b) = \sqrt{ab}$, Harmonic mean$=H(a,b) = \frac{2ab}{a+b}$, Contra harmonic mean=$C(a,b) = \frac{a^2+b^2}{a+b}$ and Power mean=$M_r(a,b) = \left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}}$ andif $r \neq 0$ have their own importance in the literature see ([1]-[6], [11, 12, 14, 17]).

The Heron mean denoted by

$$H_e(a,b) = \frac{a+\sqrt{ab}+b}{3} \tag{1}$$

Occurs in an Egyptian manuscript in the year 1850 B.C. The Moscow Papyrus presented the *volume* in terms of Heron mean in the form of $V = hH_e(a,b)$, where $H_e(a,b)$ is the Heron mean of a and b, h is the height of the frustum of a pyramid, a is the lower base area and b is the upper base area, for details see [1].

Zhen-Gang Xiao et al. [19] and various other authors have obtained some interesting and valuable results on generalization of Heron mean $H_e(a,b) = \frac{a+\sqrt{ab}+b}{3}$, using the generalized "Vander Monde's determinants". These type of generalizations and applications have generated an impressive amount of work in this field.

In [3], the generalised Power type Heron mean is define as below:

$$H_e(a,b;k) = \begin{cases} \left(\frac{a^k+(ab)^{\frac{k}{2}}+b^k}{3}\right)^{\frac{1}{k}} & \text{if } k \neq 0 \\ \sqrt{ab} & \text{if } k = 0. \end{cases} \tag{2}$$

The basic results needed to develop this paper that is convexity of one function with respect to another function were discussed in [1] and also some convexity results on various important means and their applications to mean inequalities were found in [13, 15, 16, 18].

2 Definitions and Lemmas

In this section, recall some definitions and lemmas necessary to develop this paper.

Definition 2.1. *[17] A mean is defined as a function*

$$M : R_+^2 \to R_+,$$

which has the property

$$a \wedge b \leq M(a,b) \leq a \vee b, \forall a,b > 0,$$

where

$$a \wedge b = min(a,b) \quad and \quad a \vee b = max(a,b).$$

The following lemmas are stated by using "Vander Monde's determinant" for some continuous well known functions.

Lemma 2.1. *For $\varphi(x) = x^2$ and $a = (a,b,c)$ is just the determinant of Vander Monde's matrix of the 3^{rd} order takes the form:*

$$V(a;\varphi = x^2) = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \tag{3}$$

which is equivalently

$$V(a;q=2,p=0)=(b-a)(c-a)(c-b). \tag{4}$$

Lemma 2.2. *For $\varphi(x)=x^s$ and $a^{1/2}=(\sqrt{a},\sqrt{b},\sqrt{c})\in R^3$ is just the determinant of "Vander Monde's matrix" of the 3^{rd} order takes the form:*

$$V(a;q=s-2,p=0)=V(a;\varphi=x^s)=\begin{vmatrix}1 & \sqrt{a} & a^s\\ 1 & \sqrt{b} & b^s\\ 1 & \sqrt{c} & c^s\end{vmatrix}=\begin{vmatrix}1 & \sqrt{a} & a^s\\ 0 & \sqrt{b}-\sqrt{a} & (b^s-a^s)\\ 0 & \sqrt{c}-\sqrt{a} & (c^s-a^s)\end{vmatrix} \tag{5}$$

which is equivalently

$$V(a;q=s-2,p=0)=[(c^s-a^s)(\sqrt{b}-\sqrt{a})-(b^s-a^s)(\sqrt{c}-\sqrt{a})]. \tag{6}$$

Lemma 2.3. *For $\varphi(x)=x^{1/2}=\sqrt{x}$ and $a=(a,b,c)$ is just the determinant of Van Der Monde's matrix of the 3^{rd} order takes the form:*

$$V(a;q=-3/2,p=0)=V(a;\varphi=\sqrt{x})=\begin{vmatrix}1 & a & \sqrt{a}\\ 1 & b & \sqrt{b}\\ 1 & c & \sqrt{c}\end{vmatrix} \tag{7}$$

which is equivalently

$$V(a;\varphi=\sqrt{x})=(\sqrt{b}-\sqrt{a})(\sqrt{c}-\sqrt{a})(\sqrt{b}-\sqrt{c}). \tag{8}$$

Lemma 2.4. *For $\varphi(x)=x^s$ and $a=(a^{s/2},b^{s/2},c^{s/2})\in R^3$ is just the determinant of "Vander Monde's matrix" of the 3^{rd}order order takes the form:*

$$V(a^{s/2};\varphi(x)=x^s)=\begin{vmatrix}1 & a^{s/2} & a^s\\ 0 & b^{s/2}-a^{s/2} & b^s-a^s\\ 0 & c^{s/2}-a^{s/2} & c^s-a^s\end{vmatrix}\geq 0 \tag{9}$$

which is equivalently

$$\left[(c^{s/2}-a^{s/2})(b^{s/2}-a^{s/2})(c^{s/2}-b^{s/2})\right]\geq 0. \tag{10}$$

Lemma 2.5. *Let $f(x)$ and $g(x)$ are two functions, then $g(x)$ is said to be convex with respect to $f(x)$ for $a\leq b\leq c$ if and only if*

$$\begin{vmatrix}1 & f(a) & g(a)\\ 1 & f(b) & g(b)\\ 1 & f(c) & g(c)\end{vmatrix}\geq 0 \tag{11}$$

which is equivalently

$$\begin{vmatrix}1 & f(a) & g(a)\\ 0 & f(b)-f(a) & g(b)-g(a)\\ 0 & f(c)-f(a) & g(c)-g(a)\end{vmatrix}\geq 0 \tag{12}$$

or

$$([f(b)-f(a)][g(c)-g(a)]-[f(c)-f(a)][g(b)-g(a)])\geq 0. \tag{13}$$

Setting $a=x$ and $b=1$ in Arithmetic mean=$A(x,1)=\frac{x+1}{2}$, Geometric mean=$G(x,1)=\sqrt{x}$, Harmonic mean=$H(x,1)=\frac{2x}{x+1}$, Contra harmonic mean=$C(x,1)=\frac{x^2+1}{x+1}$, Heron mean=$H_e(x,1)=\frac{x+\sqrt{x}+1}{3}$, Power mean=$M_r(x,1)=\left(\frac{x^r+1}{2}\right)^{\frac{1}{r}}$ and power type Heron mean=$H_e(x,1;k)=\left(\frac{x^k+x^{\frac{k}{2}}+1}{3}\right)^{\frac{1}{k}}$.

3 Main Result

In this section, the necessary and sufficient conditions for convexity and concavity of the k^{th} power of Power type Heron mean with respect to r^{th} power of Power mean are proved.

Theorem 1. *Let $0 < a < b < c$, then*

1. *For $k \neq r$, the r^{th} power of Power mean is convex with respect to k^{th} power of Power type Heron mean if $r > k$ and concave if $r < k/2$ and the k^{th} power of Power type Heron mean is convex with respect to r^{th} power of Power mean if $r > k$ and concave if $r < k/2$*

2. *For $k = r$, the k^{th} power of Power type Heron mean is convex with respect to r^{th} power of Power mean if $k \geq 0$ and concave if $k \leq 0$.*

Proof: Consider the k^{th} power of Power type Heron mean and r^{th} Power of Power mean in the form

$$H_e^k(x,1;k) = \left(\frac{x^k + x^{\frac{k}{2}} + 1}{3}\right) \quad \text{and} \quad M_r^r(x,1) = \left(\frac{x^r+1}{2}\right)$$

Let

$$f(x) = \left(\frac{x^k + x^{\frac{k}{2}} + 1}{3}\right) \qquad \text{and} \quad g(x) = \left(\frac{x^r+1}{2}\right)$$

then by lemma (2.5), leads to

$$\begin{vmatrix} 1 & f(a) & g(a) \\ 0 & f(b)-f(a) & g(b)-g(a) \\ 0 & f(c)-f(a) & g(c)-g(a) \end{vmatrix} = \begin{vmatrix} 1 & (a^k + a^{k/2}+1)/3 & (a^r+1)/2 \\ 0 & (b^k + b^{k/2} - a^k - a^{k/2})/3 & (b^r - a^r)/2 \\ 0 & (c^k + c^{k/2} - a^k - a^{k/2})/3 & (c^r - a^r)/2 \end{vmatrix} \tag{14}$$

proof of (1), for $k \neq r$ on simplifying the above determinant becomes $= \frac{1}{6}[(c^r - a^r)\{b^k - a^k + b^{k/2} - a^{k/2}\} - (b^r - a^r)\{c^k - a^k + c^{k/2} - a^{k/2}\}]$

Let $C^r = c^r - a^r, B^r = b^r - a^r$, the above relation becomes

$= \frac{1}{6}[C^r(B^k + B^{k/2}) - B^k(C^r + C^{k/2})]$

$= \frac{1}{6}[\triangle]$

For $0 < a < b < c$, we have

$$C^r > B^r \quad \textit{and} \quad C^k + C^{k/2} > B^k + B^{k/2}$$

holds for all positive real values of r and k.
Let us discuss the monotonicity of $\triangle$
If $\triangle < 0$, then

$$C^r(B^k + B^{k/2}) < B^r(C^k + C^{k/2})$$

On simplification the above inequality becomes

$$(B^{k-r}+B^{k/2-r})<(C^{k-r}+C^{k/2-r})$$

The inequality holds for

$$k-r>0 \quad and \quad k/2-r>0$$

equivalently holds for

$$Min(k,k/2)>r$$

This proves that $\triangle<0$ for $r<k/2$.

Thus we conclude that the k^{th} power of Power type Heron mean is concave with respect to r^{th} power of Power mean for all real values of $r<k/2$.

Let $Max(k,k/2)<r$
equivalently $\quad k<r \quad or \quad k/2<r$
$\quad k-r<0 \quad or \quad k/2-r<0$

then, $\quad C^{k-r}<B^{k-r}$ and $\quad C^{\frac{k}{2-r}}<B^{\frac{k}{2-r}}$

On simple manipulation leads to $B^k[C^k+C^{k/2}]<C^k[B^k+C^{k/2}]$

This proves that $\triangle>0 \qquad for \qquad r>k.$

Thus we conclude that the r^{th} power of Power mean is convex with respect to k^{th} power of Power type Heron mean. proof of (2), for $k=r$, the eqn (3.1) becomes

$$\begin{vmatrix} 1 & \frac{a^k+a^{k/2}+1}{3} & \frac{a^r+1}{2} \\ 0 & b^k+b^{k/2}-a^k-a^{k/2} & b^k-a^k \\ 0 & c^k+c^{k/2}-a^k-a^{k/2} & c^k-a^k \end{vmatrix}$$

on simplifying the determinant leads to $=\frac{1}{6}[(c^{k/2}-a^{k/2})(b^{k/2}-a^{k/2})(c^{k/2}-a^{k/2}-b^{k/2}+a^{k/2})]$
then by lemma (2.4) we have

$$=\frac{1}{6}[(c^{k/2}-a^{k/2})(b^{k/2}-a^{k/2})(c^{k/2}-b^{k/2})]\geq 0.$$

Thus the proof of theorem (1) completes.

In perticular, $k=r=1$, $0<a<b<c$, then the k^{th} power of Power type Heron mean in concave/convex with respect to r^{th} power of Power mean if $V(a;q=0,p=0)\geq(\leq)0$.

$$\frac{1}{6}[(\sqrt{c}-\sqrt{a})(\sqrt{b}-\sqrt{a})(\sqrt{c}-\sqrt{b})]\geq 0 \tag{15}$$

Theorem 2. *The Heron mean is concave(convex) with respect to Arithmetic mean if and only if* $V(a;r=1,p=1)\geq(\leq)0$.

Proof: Consider the Heron mean and Arithmetic mean in the form

$$H_e(x,1)=\frac{x+\sqrt{x}+1}{3} \quad \text{and} \quad A(x,1)=\frac{x+1}{2}$$

Let

$$f(x)=\frac{x+\sqrt{x}+1}{3} \quad \text{and} \quad g(x)=\frac{x+1}{2}$$

then by lemma (2.5)we have

$$\begin{vmatrix} 1 & f(a) & g(a) \\ 0 & f(b)-f(a) & g(b)-g(a) \\ 0 & f(c)-f(a) & g(c)-g(a) \end{vmatrix} = \begin{vmatrix} 1 & \frac{a+\sqrt{a}+1}{3} & \frac{a+1}{2} \\ 0 & \frac{b+\sqrt{b}-a-\sqrt{a}}{3} & \frac{b-a}{2} \\ 0 & \frac{c+\sqrt{c}-a-\sqrt{a}}{3} & \frac{c-a}{2} \end{vmatrix}$$

on simplifying the determinant leads to

$$\frac{1}{6}(\sqrt{c}-\sqrt{a})(\sqrt{b}-\sqrt{a})(\sqrt{b}-\sqrt{c}) \geq 0$$

then by lemma (2.3),

$$(\sqrt{c}-\sqrt{a})(\sqrt{b}-\sqrt{a})(\sqrt{b}-\sqrt{c}) = V(a; q=1, p=1) \geq 0. \tag{16}$$

similarly by considering

$$f(x)=\frac{x+1}{2} \quad \text{and} \quad g(x)=\frac{x+\sqrt{x}+1}{3}$$

then by lemma (2.5) we have

$$(\sqrt{c}-\sqrt{a})(\sqrt{b}-\sqrt{a})(\sqrt{c}-\sqrt{b}) = V(a; q=1, p=1) \geq 0. \tag{17}$$

By combining eqs (3.3) and (3.4), the proof of theorem (2) completes.

with similar arguments follows the proof of the following theorems.

Theorem 3. *The Heron mean is concave(convex) with respect to Geometric mean if and only if* $V(a; q=0, p=1) \geq (\leq) 0$.

Theorem 4. *The Heron mean is concave(convex) with respect to Harmonic mean if and only if* $V(a; q=-1, p=1) \geq (\leq) 0$.

Theorem 5. *The geometric mean is concave(convex) with respect to arithmetic mean if and only if* $V(a; q=1, p=0) \leq (\geq) 0$.

Theorem 6. *The Geometric mean is concave(convex) with respect to Harmonic mean if and only if* $V(a; q=-1, p=0) \geq (\leq) 0$.

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