Some Properties of *- n-class Q Operators

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Abstract

In this paper, we introduce a new class of operators, which we call the class of *-n-class Q operators. This class of operators contains the class of * paranormal operators. We prove some basic properties and a structure theorem for *-n-class Q operators. We also give the matrix representation for this class of operators.

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1 Introduction and preliminaries

Let $H$ be an infinite dimensional separable complex Hilbert space. Let $B(H)$ be the algebra of all bounded linear operators acting on $H$. Let $T$ be an operator on $H$. An operator $T$ is called class Q [2], if

$$T^*T^2 - 2T^*T + I \geq 0,$$

equivalently $T \in Q$ if $\|Tx\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2)$ for every $x \in H$. It was also proved that $T$ is paranormal if and only if $\lambda T$ is in class Q for all $\lambda > 0$ and every paranormal operator is a normaloid of class Q. Also he showed that the restriction of $T$ to an invariant subspace is again a class Q operator.

Devika, Suresh [1], introduced a new class of operators which we call the quasi class Q operators and it is defined as for $T \in B(H)$,

$$\|T^2x\|^2 \leq \frac{1}{2}(\|T^3x\|^2 + \|Tx\|^2),$$

for every $x \in H$. A k-quasi class Q operator is defined as follows [3]. An operator $T$ is said to be k-quasi class Q operator if
\[ \|T^{k+1}x\|^2 \leq \frac{1}{2}(\|T^{k+2}x\|^2 + \|T^kx\|^2), \]

for every \( x \in H \) and \( k \) is a natural number. D. Senthilkumar, prasad T in [4], has defined the new class of operators, which we call \( M \)-class Q operators. An operator \( T \) is called \( M \)-class Q if for a fixed real number \( M \geq 1 \) \( T \) satisfies

\[ M^2T^*T^2 - 2T^*T + I \geq 0, \]

or equivalently \( \|Tx\|^2 \leq \frac{1}{2}(M^2\|T^2x\|^2 + \|x\|^2) \) for every \( x \in H \) and for a fixed real number \( M \geq 1 \). In [6], Youngoh Yang and Cheoul jun Kim introduced a class Q* operators if

\[ T^*T^2 - 2TT^* + I \geq 0, \]

then \( T \) is called class Q* operators. He also proved that if \( T \) is class Q* if and only if \( \|T^*x\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2) \) for every \( x \in H \). Authors have introduced classes of quasi class Q*, k-quasi class Q*, M-class Q* and studied properties of these classes of operators.

If \( T \in B(H) \), we shall write \( N(T) \) and \( R(T) \) for the null space and the range of \( T \), respectively. Also, let \( \sigma(T) \) and \( \sigma_a(T) \) denote the spectrum and the approximate point spectrum of \( T \), respectively. Let \( \sigma_p(T), \pi(T) \), \( E(T) \) denotes the point spectrum of \( T \), the set of poles of the resolvent of \( T \), the set of all eigenvalues of \( T \) which are isolated in \( \sigma(T) \), respectively. The spectrum \( \sigma(T) \) of \( T \) is the set of \( \lambda \) such that \( T - \lambda \) is not invertible on all of the Hilbert space, where the \( \lambda \)'s are complex numbers and \( I \) is the identity operator.

In this paper, we introduce a new class of operators, which we call the class of *-\( n \)-class Q operators. This class of operators contains the class of * paranormal operators. We prove some basic properties and a structure theorem for *-\( n \)-class Q operators. We also give the matrix representation for this class of operators and we prove that the restriction of *-\( n \)-class Q operator \( T \) to an invariant subspace is again *-\( n \)-class Q operator.

### 2 Main results

In this section we have defined and give some properties of *-\( n \)-class Q operators.

**Definition 1.** An operator \( T \in B(H) \) is said to be *-\( n \)-class Q operator if for every positive integer \( n \geq 2 \) and for every \( x \in H \)

\[ \|T^*x\|^2 \leq \frac{1}{1+n}(\|T^{1+n}x\|^2 + n\|x\|^2). \]

**Theorem 2.** For each positive integer \( n \geq 2 \), \( T \) is *-\( n \)-class Q operator if and only if \( T^{*1+n}T^{1+n} - (1 + n)TT^* + nI \geq 0 \).

**Proof.** Since \( T \) is *-\( n \)-class Q operator, then

\[ \|T^*x\|^2 \leq \frac{1}{1+n}(\|T^{1+n}x\|^2 + n\|x\|^2) \]
for every $x \in H$ and for any positive integer $n \geq 0$. Then
\[
\|T^{1+n}x\|^2 + n\|x\|^2 \geq (1 + n)\|T^*x\|^2
\]
\[
\|T^{1+n}x\|^2 - (1 + n)\|T^*x\|^2 + n\|x\|^2 \geq 0
\]
\[
\langle T^{1+n}x, T^{1+n}x \rangle - (1 + n)\langle T^*x, T^*x \rangle + n\langle x, x \rangle \geq 0
\]
\[
\langle (T^{1+n}T^{1+n} - (1 + n)TT^* + n)x, x \rangle \geq 0
\]
\[
T^{*1+n}T^{1+n} - (1 + n)TT^* + nI \geq 0
\]
\[
\square
\]

**Theorem 3.** If $T \in B(H)$ is *-class Q operator then $T$ is *-$n$-class Q operator.

**Proof.** Let $T \in B(H)$ is *-class Q operator, then $T^2T^2 \geq 2TT^* - I$. We use induction principle, when $n = 1$, the result is true. When $n = 2$, $T^3T^3 - (3)TT^* + 2I \geq T^*(2TT^* - I)T - 3TT^* + 2 \geq 0$. Now we assume the result is true for $n = k - 1$. Then for $n = k$, $T^{*1+k}T^{1+k} - (1 + k)TT^* + kI \geq 0 \quad \square$

**Corollary 4.** If $T \in B(H)$ is *-$n$ class Q operator then $T$ is *-$n + 1$-class Q operator.

**Corollary 5.** If $T \in B(H)$ is *-$n$ class Q operator then $\alpha T$ is also *-$n$-class Q operator.

**Theorem 6.** Let $T \in B(H)$. If $\lambda \frac{1}{\lambda} T$ is an operator of *-$n$-class Q, then $T$ is *-$n$-paranormal operator for all $\lambda > 0$.

**Proof.** Since $\lambda \frac{1}{\lambda} T$ is an operator of *-$n$-class Q then
\[
(\lambda \frac{1}{\lambda} T)^{(1+n)}(\lambda \frac{1}{\lambda} T)^{1+n} - (1 + n)(\lambda \frac{1}{\lambda} T)(\lambda \frac{1}{\lambda} T)^* + nI \geq 0
\]
By multiplying $|\lambda|^{1+n}$ and letting $|\lambda| = \mu$, we have $T$ is *-$n$-paranormal operator for all $\lambda > 0. \quad \square$

By simple calculation we get the following results.

**Theorem 7.** If *-$n$-class Q operator $T$ doubly commutes with an isometric operator $S$, then $TS$ is an operator of *-$n$-class Q.

**Theorem 8.** If a *-$n$-class Q operator $T \in B(H)$ is unitarily equivalent to operator $S$, then $S$ is an operator of *-$n$-class Q.

**Theorem 9.** If $T \in B(H)$ is of *-$n$ class Q operator for a positive integer $n$, $0 \neq \lambda \in \sigma_p(T)$ and $T$ is of the form $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \ker(T - \lambda) \oplus \text{ran}(T - \lambda)$, then 1. $T_2 = 0$ and 2. $T_3$ is *-$n$-class Q operator.

**Proof.** Let $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \ker(T - \lambda) \oplus \text{ran}(T - \lambda)$. Without the loss of generality assume that $\lambda = 1$, then by Theorem 2, $T^{*1+n}T^{1+n} - (1 + n)TT^* + nI \geq 0$. 

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Now,
\[ T^{1+n} = \left( \frac{1}{0} \sum_{j=0}^{n} T_2 T_3^{n-j} \right) \] and
\[ T^{*1+n} = \left( \frac{1}{0} \sum_{j=0}^{n} T_2 T_3^{n-j} T_3^{*1+n} \right) \]
\[ T^{*1+n} T^{1+n} = \left( \frac{1}{0} \sum_{j=0}^{n} T_2 T_3^{n-j} \right) \left( \frac{1}{0} \sum_{j=0}^{n} T_2 T_3^{n-j} \right) T_3^{*1+n} \]
So, \( T^{*1+n} T^{1+n} - (1 + n)TT^* + nI \geq 0 \) gives
\[ \left( \begin{array}{cc} A & B \\ B^* & C \end{array} \right) \geq 0 \]
Where \( A = 1 - (1 + n)(1 + T_2 T_2^*) + n, \ B = \sum_{j=0}^{n} T_2 T_3^{n-j} - (1 + n)T_2 T_3^* \) and \( C = (\sum_{j=0}^{n} T_2 T_3^{n-j})^* \sum_{j=0}^{n} T_2 T_3^{n-j} + T_3^{*1+n} T_3^{1+n} - (1 + n)T_3 T_3^* + n \)
But, we know that, ” If \( A \) is a matrix of the form \( \left( \begin{array}{cc} A & B \\ B^* & C \end{array} \right) \geq 0 \) if and only if \( A \geq 0, C \geq 0 \) and \( B = A^2 WC^2 \) for some contraction \( W \).
Therefore \( 1 - (1 + n)(1 + T_2 T_2^*) + n \geq 0 \), which implies that \(- (1 + n)T_2 T_2^* \geq 0 \). This gives \( T_2 = 0, \) since \( n \) is a positive integer. Also \( T_3 \) is \(*-n\)-class Q operator.

**Corollary 10.** If \( T \in B(H) \) is of \(*-n\) class Q operator for a positive integer \( n \), then \( T \) is of the form \( T = \left( \begin{array}{cc} \lambda & 0 \\ 0 & T_3 \end{array} \right) \) on \( H = \ker(T - \lambda) \oplus \overline{\text{ran}(T - \lambda)} \), where \( T_3 \) is \(*-n\) class Q operator and \( \ker(T - \lambda) = \{0\} \).

**Theorem 11.** If \( T \in B(H) \) is a \(*-n\)-class Q operator for a positive integer \( n \), \( T \) does not have dense range and \( T \) has the following \( 2 \times 2 \) matrix representation
\[ T = \left( \begin{array}{cc} T_1 & T_2 \\ 0 & T_3 \end{array} \right) \] on \( H = \overline{\text{ran}(T)} \oplus \ker T^* \), if and only if \( T_1 \) is also \(*-n\)-class Q operator on \( \overline{\text{ran}(T)} \) and \( T_3 = 0 \). Further more \( \sigma(T) = \sigma(T_1) \cup \{0\} \) where \( \sigma(T) \) denotes the spectrum of \( T \).

**Proof.** Let \( T \in B(H) \) be \(*-n\) class Q operator and \( P \) be an orthogonal projection onto \( \overline{\text{ran}(T)} \). Then \( T_1 = TP = PTP \). By Theorem 2 we have that
\[ P(T^{*1+n} T^{1+n} - (1 + n)TT^* + nI)P \geq 0 \]
\[ T^{*1+n} T^{1+n} - (1 + n)(T_1 T_1^*) + nI \geq (1 + n)T_2 T_2^* \]
\[ \geq (1 + n)|T_2|^2 \geq 0 \]
Therefore \( T_1 \) is \(*-n\)-class Q operator on \( \overline{\text{ran}(T)} \). Also for any \( x = (x_1, x_2) \in H \),
\[ \langle T_3^k x_2, x_2 \rangle = \langle T^k(I - P)x, (I - P)x \rangle \]
\[ = \langle (I - P)x, T^*k(I - P)x \rangle = 0 \]
This implies \( T_3 = 0 \)

Since \( \sigma(T) \cup \tau = \sigma(T_1) \cup \sigma(T_3) \) where \( \tau \) is the union of the holes in \( \sigma(T) \), which happens to be a subset of \( \sigma(T_1) \cap \sigma(T_3) \) [by corollary 7, 11]. \( \sigma(T_3) = 0 \) and 
\( \sigma(T_1) \cap \sigma(T_3) \) has no interior points we have \( \sigma(T) = \sigma(T_1) \cup \{0\} \).

Suppose that \( T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \) on \( H = \text{ran}(T) \oplus \ker T^* \) where \( T_1 \) is \( *\)-class \( Q \) operator on \( \text{ran}(T) \) and \( T_3 = 0 \)

\[
T^{1+n} = \begin{pmatrix} T_1^{1+n} & \sum_{j=0}^{n} T_j^j T_2^j T_3^{n-j} \\ 0 & T_3^{1+n} \end{pmatrix}
\]

\[
T^{*1+n} = \begin{pmatrix} T_1^{*1+n} & (\sum_{j=0}^{n} T_j^j T_2^j T_3^{n-j})^* \\ 0 & T_3^{*1+n} \end{pmatrix}
\]

\[
T^{*1+n} T^{1+n} = \begin{pmatrix} T_1^{*1+n} T_1^{1+n} & T_1^{*1+n} \sum_{j=0}^{n} T_j^j T_2^j T_3^{n-j} \\ (\sum_{j=0}^{n} T_j^j T_2^j T_3^{n-j})^* T_1^{1+n} & (\sum_{j=0}^{n} T_j^j T_2^j T_3^{n-j})^* (\sum_{j=0}^{n} T_j^j T_2^j T_3^{n-j}) + T_3^{1+n} T_3^{*1+n} \end{pmatrix}
\]

Since \( T_3 = 0 \) and \( T \) is \( *\)-class \( Q \) operator,

\[
T^{*1+n} T^{1+n} - (1+n)TT^* + nI = \begin{pmatrix} T_1^{*1+n} T_1^{1+n} - (1+n)(T_1 T_1^* + T_2 T_2^*) + n & 0 \\ 0 & 0 \end{pmatrix} \geq 0
\]

Hence \( T \) is \( *\)-class \( Q \) operator. \( \Box \)

**Theorem 12.** Let \( M \) be a closed \( T \)-invariant subspace of \( H \). Then the restriction \( T|_M \) of a \( *\)-class \( Q \) operator \( T \) to \( M \) is \( *\)-class \( Q \) operator.

**Proof.** By Theorem 11, \( T|_M \) is also \( *\)-class \( Q \) operator. \( \Box \)

**Theorem 13.** If \( T \) is \( *\)-class \( Q \) operator and \( \lambda \neq 0 \), then \( Tx = \lambda x \) implies \( T^*x = \bar{\lambda}x \) for every unit vector \( x \) in \( H \).

**Proof.** Suppose \( T \) is \( *\)-class \( Q \) operator and since \( Tx = \lambda x \), we have \( |\lambda|^2 \leq \frac{1}{1+n}(|\lambda|^{2(1+n)} + n) \geq |\lambda|^2 \), then \( \frac{1}{1+n}(|\lambda|^{2(1+n)} + n) = |\lambda|^2 \)

Hence \( \|T^*x\|^2 \leq |\lambda|^2 \). Also \( \langle (T - \lambda)x, (T - \lambda)^*x \rangle \leq |\lambda|^2 - 2|\lambda|^2 + |\lambda|^2 = 0 \). Hence, \( T^*x = \bar{\lambda}x \) \( \Box \)

**Corollary 14.** Let \( T \) is \( *\)-class \( Q \) operator and \( \lambda, \mu \) be distinct eigen values of \( T \). If \( x \) and \( y \) are the corresponding eigen vectors of \( \lambda \) and \( \mu \) respectively, then \( \langle x, y \rangle = 0 \).

**Corollary 15.** If \( T^* \) is \( *\)-class \( Q \) operator then \( \beta(T - \lambda) \leq \alpha(T - \lambda) \) for all \( \lambda \in C \).

**Corollary 16.** For \( T \in B(H) \), let \( T \) is \( *\)-class \( Q \) operator.

1. If \( \lambda \in \sigma_a(T) \) and \( \| (T - \lambda)x_m \| \to 0 \) for unit vectors \( x_m \) then \( \| (T - \lambda)^*x_m \| \to 0 \)

2. Let \( \lambda \) and \( \mu \) (\( \lambda \neq \mu \)) be in \( \sigma_a(T) \). If \( \| (T - \lambda)x_m \| \to 0 \) and \( \| (T - \mu)y_m \| \to 0 \) for unit vectors \( x_m \) and \( y_m \) then \( \langle x_m, y_m \rangle \to 0 \).
Theorem 17. Let $T$ be a regular *- $n$ class Q operator, then the approximate point spectrum lies in the disc

$$\sigma_{ap}(T) \subseteq \{ \lambda \in \mathbb{C} : \frac{(1+n)^{1/2}}{||T^{-1}||(||T'^n||^2+n||T^{-1}||^2)^{1/2}} \leq |\lambda| \leq ||T|| \}$$

Proof. Suppose $T$ is regular $n$ class Q operator, then for every unit vector $x$ in $H$, we have

$$\|x\|^2 = \|T^{-1}T^*x\|^2 \leq \|T^{-1}\|^2\|T^*x\|^2$$

$$\leq \|T^{-1}\|^2\left(\frac{1}{1+n}(\|T^{1+n}x\|^2 + n\|x\|^2)\right)$$

$$\leq \|T^{-1}\|^2\left(\|T^n\|^2\|Tx\|^2 + n\|T^{-1}\|^2\|Tx\|^2\right)$$

Hence $\|Tx\|^2 \geq \frac{(1+n)\|x\|^2}{\|T^{-1}\|^2(\|T^n\|^2 + n\|T^{-1}\|^2)}$

Now assume that $\lambda \in \sigma_{ap}(T)$. Then there exists a sequence $\{x_m\}$, $\|x_m\| = 1$ such that $\|(T - \lambda x_m)\| \to 0$ when $m \to \infty$ we have

$$\|Tx_m - \lambda x_m\| \geq \|Tx_m\| - |\lambda||x_m||$$

$$\geq \|T\| - |\lambda|$$

$$\geq \frac{(1+n)^{1/2}}{\|T^{-1}\|^2(\|T^n\|^2 + n\|T^{-1}\|^2)^{1/2}} - |\lambda|$$

Now when $m \to \infty$, $|\lambda| \geq \frac{(1+n)^{1/2}}{\|T^{-1}\|^2(\|T^n\|^2 + n\|T^{-1}\|^2)^{1/2}}$ \hfill \square

References


