A Study on Cost Analysis of a Multi Server Fuzzy Queueing- Inventory Model With Waiting and Service Costs

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Abstract

This paper consider a study on waiting cost and service cost of a multi-server fuzzy queueing-inventory model. This model developed the effect of queueing in relation to the time spent by customers. Waiting lines and service systems are important parts of the business world. The model illustrated for customers on a level with service is the multiple-channel queuing model with Poisson Arrival and Exponential Service Times. The eye shape fuzzy number is defined and its properties are given. Our proposed model have been considered in fuzzy environment. The parameters involved in this model are represented by eye shape fuzzy number. The steady state probability distribution and the other performance measures are derived by using the inventory level state transitions. The total expected cost is defuzzified by the Generalised weighted canonical representation of eye shape fuzzy number. Then by using the MATLAB software the total expected cost, reorder level, waiting cost and service cost are found in both crisp and fuzzy.

Keywords:- Queueing, Inventory, Eye shape fuzzy number, Generalised weighted canonical representation technique.

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1 INTRODUCTION

The inventory modelling is also very important because the decision making affects the total business. i.e., input and output of products minimization of cost. Research on queueing - inventory systems has captured much attention over the last decades. A queueing inventory system is different from the traditional queueing system because of the way the attached inventory influences the service. If there is no inventory
Berman and Kim [5] analyzed a queueing inventory system with Poisson arrivals, exponential service times and zero lead times. The authors proved that the optimal policy is never to order when the system is empty. Berman and Sapna [4] studied queueing-inventory systems with Poisson arrivals, arbitrary distribution service times and zero lead times.

The optimal value of the maximum allowable inventory which minimizes the long-run expected cost rate has been obtained. Berman and Sapna [4] discussed a finite capacity system with Poisson arrivals, exponential distributed lead times and service times. The existence of a stationary optimal service policy has been proved. [4] studied internet-based supply chains with Poisson arrivals, exponential service times and the Erlang lead times and found that the optimal ordering policy has a monotonic threshold structure. [5] addressed an infinite capacity queueing-inventory system with Poisson arrivals, exponential service times and exponential lead times. The authors identified a replenishment policy which maximized the system profit.

Schwarz et al. [4] derived stationary distributions of joint queue length and inventory processes in explicit product form for M/M/1 queueing inventory system with lost sales under various inventory management policies such as (r,Q) policy and (r,S) policy. The M/M/1 queueing-inventory system with backordering was investigated by Schwarz and Daduna [8]. Heetal. [9] quantified the value of information used in inventory control.

A common situation that occurs in everyday life is that of queuing or waiting in line. Queues (waiting lines) are usually seen at bus stops, hospitals, bank counters and so on. Queueing system is the most applications of alternating renewal process and has been widely applied to many practical problems. In classic queueing systems, the interarrival times of customers and service times of servers are characterized as random variables, these are discussed by Gross D., [2] and Harris O. Berman [4] presented Optimal control of service for facilities holding inventory. The Kao C., Li C. C and Chen S. P., [3] are explain about the Fuzzy Queues.

In general, queues form when the demand for service exceeds its supply. Wait time depends on the number of customers (human being or objects) on queue, the number of servers serving line, and the amount of service time for each individual customer. The goal of queuing is therefore to minimize the total cost to the system. Later, the researcher has given a Queueing-Inventory system with two classes demand. This paper confer about Multi Server Fuzzy Queueing-Inventory model with waiting and service costs.

2 ASSUMPTIONS AND NOTATIONS

The following assumptions and notations are used throughout this paper.

2.1 Assumptions

1. A Multi-Server queueing inventory system.

2. Multiple classes of customers getting service in Multiserver.

3. The arrival process for Multiple classes is state independent.
4. Customers arrival follows poisson process.
5. Service discipline FCFS (First come first service)
6. Lead-time is exponentially distributed with parameter $\mu$.

2.2 Notations

Notations:
- $Q$ - Ordering Quantity,
- $s$ - Reorder point,
- $I(t)$ - on-hand inventory level at time $t$,
- $\lambda_i$ - the arrival rate for the multiple classes of customers,
- $\mu_i$ - the service rate for the multiple classes of customers,
- $S$ - the number of servers,
- $\rho_0$ - the probability that there are no customers in the system,
- $Lq$ - Expected number of customers in the queue,
- $Ls$ - Expected number of customers in the system,
- $Wq$ - Expected time a customer spends in the queue,
- $Ws$ - Expected time a customer spends in the system.
- $Cs$ - service cost of each server

3. MATHEMATICAL MODEL IN CRISP ENVIRONMENT:

3.1 Problem Formulation and Analysis

Consider an inventory system in which there are multiple classes of multiple service. The arrival process for multiple classes of customers is state independent and each customer needs exactly one item from the inventory. The arrival rate of Multi classes of customers follows a poisson process with $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ when the on hand inventory drops to a fixed level $s$, an order for $Q+s$ units is placed. Let $I(t)$ denote the on-hand inventory level at time $t$. Since $Q(s)$, at any given point of time there is almost one order pending, and as such from our assumptions it is clear that the $E=0,1,2,\ldots,Q+s$ is a markov process. Let $P(i,j,t) = \text{Pr}[I(t) = j/I(0) = i]$, $i,j \in E$. In the steady state, let $P(j) = \text{limit} P(i,j,t)$. Then $P(j)$ satisfies the following balance equations.

\[
\left(\sum_{i=1}^{n} \lambda_i\right) P(Q + s) = \left(\sum_{i=1}^{n} \mu_i\right) P(s) \quad (1)
\]

\[
\left(\sum_{i=1}^{n} \lambda_i\right) P(j) = \left(\sum_{i=1}^{n} \lambda_i\right) P(j + 1) + P(j - Q), \quad j = Q, Q + 1, \ldots, Q + s - 1 \quad (2)
\]

\[
\left(\sum_{i=1}^{n} \lambda_i\right) P(j) = \left(\sum_{i=1}^{n} \lambda_i\right) P(j + 1), \quad j = s + 1, s + 2, \ldots, Q - 1 \quad (3)
\]

\[
\left(\sum_{i=1}^{n} \lambda_i\right) + \left(\sum_{i=1}^{n} \mu_i\right) P(s) = \left(\sum_{i=1}^{n} \lambda_i\right) P(s + 1) \quad (4)
\]
Using \( \sum_{j=1}^{s+Q} j \ P(j) \)

Solve the equation by the means of recursive process and get.

\[
P(j) = \left( 1 + \frac{\sum_{i=1}^{n} \mu_i}{\sum_{i=1}^{n} \lambda_i} \right)^{j-1} \left( \frac{\sum_{i=1}^{n} \mu_i}{\sum_{i=1}^{n} \lambda_i} P(0) \right) ; j = 1, 2, \ldots, s
\]  

(7)

\[
P(j) = \left( 1 + \frac{\sum_{i=1}^{n} \mu_i}{\sum_{i=1}^{n} \lambda_i} \right)^{s} \left( \frac{\sum_{i=1}^{n} \mu_i}{\sum_{i=1}^{n} \lambda_i} P(0) \right) j = s + 1, s + 2, \ldots, Q
\]

(8)

\[
P(j) = \left[ \left( 1 + \frac{\sum_{i=1}^{n} \mu_i}{\sum_{i=1}^{n} \lambda_i} \right)^{s} - \left( 1 + \frac{\sum_{i=1}^{n} \mu_i}{\sum_{i=1}^{n} \lambda_i} \right)^{j-Q-1} \right] \frac{\sum_{i=1}^{n} \mu_i}{\sum_{i=1}^{n} \lambda_i} \frac{P(0)}{s} , j = s + 1, s + 2, \ldots, Q + s
\]

(9)

By solving the equations,

\[
P(0) = \frac{1}{1 + \left( \frac{\sum_{i=1}^{n} \mu_i}{\sum_{i=1}^{n} \lambda_i} \right)^{s} \left( \frac{\sum_{i=1}^{n} \mu_i}{\sum_{i=1}^{n} \lambda_i} \right) + 1}
\]

(10)

Inserting (10) in (6)-(9) respectively, we have the analytical steady state probability distributions of the inventory level. Let I denote the average inventory level.

Using \( \sum_{j=1}^{s+Q} j \ P(j) \)

\[
I = \left( 1 + \frac{\sum_{i=1}^{n} \mu_i}{\sum_{i=1}^{n} \lambda_i} \right)^{s} \ P(0) \left[ \frac{\sum_{i=1}^{n} \lambda_i}{\sum_{i=1}^{n} \mu_i} + s + \frac{(\sum_{i=1}^{n} \mu_i)Q^2 + 2Qs}{2(\sum_{i=1}^{n} \lambda_i)} \right] + hQP(0)
\]

3.2 The M/M/S Model

The model adopted in this work is the Multi-server Queuing Model. For this queuing system, it is assumed that the arrivals follow a Poisson probability distribution at an average of \( \lambda \) customers per unit time. The service times are distributed exponentially, with an average of customers per unit of time and number of servers S. If there are n customers in the queuing system at any point in time, then the following two cases may arise:

(i) If \( n<s \), (number of customers in the system is less than the number of servers), then there will be no queue. However, \( (s-n) \) number of servers will not be busy. The combined service rate will then be \( \mu_n = n\mu; n<s \).
(ii) If \( n \geq s \), (number of customers in the system is more than or equal to the number of customers in the queue will be \( (n-s) \). the combined service rate will be \( \mu_n = s\mu \); \( n \geq s \).

From the model the probability of having \( n \) customers in the system is given by

\[
P_n = \begin{cases} 
\left( \frac{s}{n!} \right) \rho^n p_0 & n < s \\
\left( \frac{s}{s!} \right) \frac{1}{s} \left( \frac{\rho_s}{\mu} \right)^n & n \geq s; \quad \rho = \frac{\lambda}{s\mu} 
\end{cases}
\]

\[
p_0 = \left[ \sum_{n=0}^{s-1} \frac{1}{n!} \left( \frac{\lambda}{\mu} \right)^n + \frac{1}{s!} \left( \frac{\lambda}{\mu} \right)^s \frac{s\mu}{s\mu - 1} \right]^{-1}
\]

We now proceed to compute the performance measures of the queueing system.

The expected number of the customer waiting on the queue is

\[
L_q = \left[ \frac{1}{(s-1)!} \left( \frac{\lambda}{\mu} \right)^s \frac{\mu\lambda}{(s\mu - 1)^2} \right] \left[ \sum_{n=0}^{s-1} \frac{1}{n!} \left( \frac{\lambda}{\mu} \right)^n + \frac{1}{s!} \left( \frac{\lambda}{\mu} \right)^s \frac{s\mu}{s\mu - 1} \right]^{-1}
\]

Expected number of customers in the system

\[
L_s = \left[ \frac{1}{(s-1)!} \left( \frac{\lambda}{\mu} \right)^s \frac{\mu\lambda}{(s\mu - 1)^2} \right] \left[ \sum_{n=0}^{s-1} \frac{1}{n!} \left( \frac{\lambda}{\mu} \right)^n + \frac{1}{s!} \left( \frac{\lambda}{\mu} \right)^s \frac{s\mu}{s\mu - 1} \right]^{-1} + \frac{\lambda}{\mu}
\]

Expected waiting time of customer in the queue

\[
W_q = \left[ \frac{1}{(s-1)!} \left( \frac{\lambda}{\mu} \right)^s \frac{\mu\lambda}{(s\mu - 1)^2} \right] \left[ \sum_{n=0}^{s-1} \frac{1}{n!} \left( \frac{\lambda}{\mu} \right)^n + \frac{1}{s!} \left( \frac{\lambda}{\mu} \right)^s \frac{s\mu}{s\mu - 1} \right]^{-1}
\]

Average time a customer spends in the system

\[
W_s = \left[ \frac{1}{(s-1)!} \left( \frac{\lambda}{\mu} \right)^s \frac{\mu\lambda}{(s\mu - 1)^2} \right] \left[ \sum_{n=0}^{s-1} \frac{1}{n!} \left( \frac{\lambda}{\mu} \right)^n + \frac{1}{s!} \left( \frac{\lambda}{\mu} \right)^s \frac{s\mu}{s\mu - 1} \right]^{-1} + \frac{\lambda}{\mu}
\]

Utilization factor the fraction of time servers are

\[
\rho = \frac{\lambda}{s\mu}
\]

4. Introducing Costs into the Model:

In order to evaluate the optimum number of servers in the system, two costs must be considered in making these decisions: (i) Service costs (ii) Waiting time costs of customers. Economic analysis of these costs helps the management to make a trade-off between the increased costs of providing better service and the decreased waiting time costs of customers derived from providing that service.

Expected Service Cost \( E(\text{SC}) = S^*Cs \)

Where, \( S^* = \) number of servers,
Expected Waiting Costs in the System E (WC) = (λ Ws )Cw

Where λ = number of arrivals,

Cw = Opportunity cost of waiting by customers we have,

Expected Total Costs E (TC) = hI + E (SC) + E (WC)

\[= h\left(1 + \left(\frac{\sum_{i=1}^{n} \mu_i}{\sum_{i=1}^{n} \mu_i}\right)s\right) P(0) \right] \left(\frac{\sum_{i=1}^{n} \lambda_i}{\sum_{i=1}^{n} \mu_i}\right) + s + \frac{\left(\sum_{i=1}^{n} \mu_i\right)Q^2 + Q + 2Qs}{2 \left(\sum_{i=1}^{n} \lambda_i\right)}\]

\[-Q + \frac{\left(\sum_{i=1}^{n} \lambda_i\right)}{\left(\sum_{i=1}^{n} \mu_i\right)} - \frac{\left(\sum_{i=1}^{n} \mu_i\right)Q}{\left(\sum_{i=1}^{n} \lambda_i\right)}\]

\[-hQP(0) + SC_s + (\lambda Ws)C_w\]

we use MATLAB software to compute the performance measures of the multi-server queueing system and the total expected cost are also found.

3 Eye Shape Fuzzy Number

Figure 1: Graphical representation of eye shape fuzzy number

Definition 1. The Eye shape fuzzy number \(\tilde{A}\) described as a normalised convex fuzzy subset on the real line \(R\) whose membership function \(\mu_{\tilde{A}(x)}\) is defined as follows

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
1 - \frac{1}{2} \left(\frac{x-a}{b-a}\right)^2 & \text{for } a \leq x \leq b \\
1 - \frac{1}{2} \left(\frac{x-b}{c-b}\right)^2 & \text{for } b \leq x \leq c \\
\frac{1}{2} \left(\frac{x-b}{b-a}\right)^2 & \text{for } a \leq x \leq b \\
\frac{1}{2} \left(\frac{x-b}{b-c}\right)^2 & \text{for } b \leq x \leq c 
\end{cases}
\]

where \(a, b\) and \(c\) are real numbers. This type of fuzzy number be denoted as \(\tilde{A} = (a,b,c)_{EFN}\) where \(\mu_{\tilde{A}}(x)\) satisfies the following conditions.

1. Opposites angles are equal in the horizontal line.
2. The horizontal and vertical diagonal bisect each other and meet at 90°.
3. In the horizontal diagonal, the base of the adjacent angles are equal.
3.1 Generalized Weighted Canonical Representation of Eye Shape Fuzzy Number

To simplify the representation of fuzzy numbers consider two parameters – value and ambiguity which represent some basic features of fuzzy numbers and hence they were called a canonical representation of fuzzy numbers.

\[ WCR_E(\tilde{A}) = w Val_E(\tilde{A}) + (1 + w) Amb_E(\tilde{A}), \text{ } W = \text{weighted value} \]

\[ WCR_E(\tilde{A}) = w \left( \frac{2b}{n+1} + \frac{(b-c)\sqrt{2}}{2} \frac{2n+3}{x} + \frac{(b-a)\sqrt{2}}{2} \frac{2n+3}{x} + \right) \]

\[ + (1 + w) \left( \frac{2(c-b)\sqrt{2}}{3} \frac{2n+3}{x} \right) \left[ \frac{(-1)^n 411n^2 - 1169n + 765}{2^n} \right] \]

\[ \left( \frac{3}{k+1} - \frac{2k+3}{x} \right) \left[ \frac{(-1)^n 411n^2 - 1169n + 765}{2^n} \right] \]

If the cost parameters are characterized by eye shape fuzzy number, then our proposed model in fuzzy environment is given by

\[ E(T\tilde{C}) = \tilde{h} \left( 1 + \frac{\sum_{i=1}^{n} \mu_i}{\sum_{i=1}^{n} \lambda_i} \right) s P(0) \left[ \frac{\sum_{i=1}^{n} \lambda_i}{\sum_{i=1}^{n} \mu_i} + s + \frac{\sum_{i=1}^{n} \mu_i Q^2 + 2Qs}{2 \left( \sum_{i=1}^{n} \lambda_i \right)} \right] \]

\[ - \sum_{i=1}^{n} \lambda_i \frac{\sum_{i=1}^{n} \mu_i}{\sum_{i=1}^{n} \lambda_i} \]

\[ + \tilde{h} Q P(0) + S\tilde{C}_s + (\lambda W s)\tilde{C}_w \]

Now by using the defuzzification technique the proposed model is reduced to

\[ = W_c(h) \left( 1 + \frac{\sum_{i=1}^{n} \mu_i}{\sum_{i=1}^{n} \lambda_i} \right) s P(0) \left[ \frac{\sum_{i=1}^{n} \lambda_i}{\sum_{i=1}^{n} \mu_i} + s + \frac{\sum_{i=1}^{n} \mu_i Q^2 + 2Qs}{2 \left( \sum_{i=1}^{n} \lambda_i \right)} \right] \]

\[ - \sum_{i=1}^{n} \lambda_i \frac{\sum_{i=1}^{n} \mu_i}{\sum_{i=1}^{n} \lambda_i} \]

\[ + W_c(h) Q P(0) + W_c(SC_s) + W_c(\lambda W s)C_w \]

we use MATLAB software to compute the performance measures of the multiserver queueing system and also the total expected cost are also found.
4 Numerical Example

Let us consider the situation where \( \lambda_1 = 3.5, \lambda_2 = 2.9, \lambda_3 = 3.9, \lambda_4 = 4.1, \lambda_5 = 4.2, \lambda_6 = 4.5, \mu_1 = 0.9, \mu_2 = 1.2, \mu_3 = 1.9, \mu_4 = 1.0, \mu_5 = 1.2, \mu_6 = 2.2, S^* = 15, \bar{h} = (10, 20, 30), \bar{c}_1 = (10, 20, 30), \bar{c}_2 = (5, 10, 15), W_s = 0.537, C_s = 100, C_w = 50 \).

<table>
<thead>
<tr>
<th>Crisp</th>
<th>( c_1 )</th>
<th>( h )</th>
<th>( c_2 )</th>
<th>( Q^* )</th>
<th>( S^* )</th>
<th>( E(\text{Total cost}) )</th>
</tr>
</thead>
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<td>20</td>
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<td>6</td>
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</tr>
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<td>10</td>
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<tr>
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<td>15</td>
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<td>7.0319</td>
<td>44.179</td>
</tr>
</tbody>
</table>

**Observation:** Table 1 shows the calculation for economic reorder quantity, reorder safety level and total cost. The total cost has been calculated in both crisp and fuzzy environment. It is observed that total expected cost is less in fuzzy while compared with crisp model and the reorder quantity and reorder level is higher in fuzzy comparing with crisp.

References


