A Study on SIS Epidemic Model using Markovian Retrial Queues

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Abstract

In this paper, we implement queuing techniques to a SIS Epidemic model. The aim of this paper is to study the epidemic model in a clear and effective manner. We extend the simple compartmental model into a the form of a retrial queue such that the model is elaborate. We intend to find a balance in the model such that the infection does not spread. We find the stationary distribution of the state variables. We derive the explicit formulae and recursive formulae for the joint stationary distribution of the number of repeated attempts and the state of the infective. We analyse the stochastic decomposition property. We also obtain a stability condition for the model.

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1 Introduction

A queue is a waiting line. A retrial queue is a queue in which a customer arriving when all servers are busy leaves the service area but after some random time repeats the request. These requests are called primary request. If an arriving primary requests finds free server, it immediately occupies this server and leaves the system after completion of service. If all servers are busy then the request is rejected and forms a source of repeated requests (retrials). The sources remain in a so called orbit. Every such source sends repeated requests until it finds a free server, in which case it is served and the source is eliminated.

We apply the queuing theory to epidemic models. We use the concept of retrial queues to construct an SIS epidemic model. Almost all the events in the society...
follow queuing patterns. We either provide service or get service. By modelling epidemic models in queuing theory, we get a more intricate look on how the system works and hence can solve the model in a more accurate manner. An SIS epidemic model is a model consisting of two populations: Infected population and susceptible population. The susceptible population is infected at a constant rate and go to infected population. Whereas the infected get recovered, but since they are susceptible of getting infected again, they go to susceptible population.

Queuing Technique for Ebola Virus Disease Transmission and Control Analysis was studied by Chinyere Ogochukwu Dike, Zaitul Marlizawati Zainuddin and Ikeme John Dike. They applied $M/M/1$ queue to ebola epidemic[4].

In this paper, we study the SIS epidemic model using Markovian retrial queues and analyse the model. Model description is given in section 2. In section 3, we provide the transition diagram and balance equations of the model. Stationary distribution is analysed in section 4. In section 5, we discuss about the stochastic decomposition of the model. Finally, conclusion of the paper is given in section 6.

2 Model Description

We consider a single server retrial queue. Here we consider that the infected individual takes the place of a server in the system. If the infected individual is recovered, he goes to state 0, if he gets infected again, he comes to state 1. Here we assume that the server can make contact with only one individual at a time $t$. $\mu$ is the rate at which the infected individual is recovered. $\lambda$ is the rate at which susceptible individual gets infected. Incoming individuals, those who are not infected but susceptible to infection approach the infective(server) according to a poisson process with a rate $\theta$. But these susceptible individuals find the infective(server) busy, that is either the infective is recovered or remains infected, will remain susceptible and may get infected in a random manner (repeat the approach in a random manner).

The susceptibles individuals those who come into contact with infective, they remain in the orbit. Those individuals from the orbit get infected at a rate $j\gamma$. In this model, we consider a single infective and $N$ susceptibles and hence it is a $M/M/1/N$ Markovian model.

3 Transition Diagram and Balance Equations

The Compartmental SIS Epidemic model is given below:

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S \xrightarrow{\beta} I
\xrightarrow{\gamma} S
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Figure 1: SIS Epidemic model

In figure 1, there are two compartments namely the susceptible population $S$ and the infected population $I$. $\beta$ is the infection rate and $\gamma$ is the recovery rate. In the compartmental model the susceptible and infected are considered as a group of population.
Our aim is to provide a closer and clear look at this epidemic model and hence make use of queuing theory. Instead of considering as a group of population, we consider the individuals. This might provide an easy and realistic approach to epidemic modelling. Let \( I(t) \) denote the state of the infective (server), that is, an individual at time \( t \geq 0 \).

\[
I(t) = \begin{cases} 
0 & \text{the individual is recovered} \\
1 & \text{the individual remains infected}
\end{cases}
\] (1)

Let \( N(t) \) denote the number of incoming susceptible individuals remaining in the orbit at time \( t \).[1],[2],[3]. We see that, \( X(t) = (I(t), N(t)); t \geq 0 \).

Figure 2: Transition diagram of SIS Epidemic Model

We assume that the Markov chain is ergodic.

\[
\pi_{i,j} = \lim_{t \to \infty} P(I(t) = i, N(t) = j), i = 0, 1, 2, 3\ldots, j \in Z_+
\]

It follows from the figure that the system of balance equations for \( \pi_{i,j} \) is given by the following difference equations[8].

\[
(\lambda + j\gamma + \theta) \pi_{0,j} = \mu \pi_{1,j} + \theta \pi_{0,j-1}
\] (2)

\[
(\mu + \theta) \pi_{1,j} = \lambda \pi_{0,j} + (j + 1)\gamma \pi_{0,j+1} + \theta \pi_{1,j-1}
\] (3)

For \( j \in Z_+ \), where \( \pi_{i,-1} = 0 \) \((i = 0, 1)\). Let \( P_i(z) \) denote the partial generating function of \( \pi_{i,j} \).

\[
P_i(z) = \sum_{j=0}^{\infty} \pi_{i,j}z^j \quad i = 0, 1, \quad |z| \leq 1
\] (4)

Multiplying the equations (2) and (3) by \( z^j \) and taking summation over \( j \), we get

\[
(\lambda + \theta)P_0(z) + \gamma z P'_0(z) = \mu P_1(z) + \theta z P_0(z)
\] (5)

\[
(\mu + \theta)P_1(z) = \lambda P_0(z) + \gamma P'_0(z) + \theta z P_1(z)
\] (6)

Summing the above two equations, we get

\[
\gamma P'_0(z)(z - 1) = \theta(z - 1)[P_0(z) + P_1(z)]
\] (7)

Dividing by \((z - 1)\) on both sides, we get

\[
\gamma P'_0(z) = \theta [P_0(z) + P_1(z)]
\] (8)
3.1 Hypergeometric Functions

The Explicit expressions in the further sections are derived using these hypergeometric functions. For a complex \( x \), \((x)_j = \begin{pmatrix} 1 \\ x(x+1)...(x+j-1) \end{pmatrix}_{j \in \mathbb{N}}^{j = 0}
\)

Denote the Pochhammer symbol, where \( N = \{1, 2, \ldots\} \). Then, for complex numbers \( a, b, c \) and \( z \), the hypergeometric function \( F(a; b; c; z) \) is defined by

\[
F(a; b; c; z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j (c)_j}{j!} z^j, \quad |z| \leq 1
\]

We can see that

\[
F(a; 1; 1; z) = \sum_{j=0}^{\infty} \frac{(a)_j}{j!} z^j = \sum_{j=0}^{\infty} \frac{(-a)(-a-1)...(-a-j+1)}{j!} (-z)^j = (1 - z)^{-a}
\]

which follows from the generalized Newton binomial formula.

4 Stationary Distribution

In this section, we get the explicit expressions for the joint stationary distribution through generating functions.

4.1 Generating Function

**Theorem 1.** The explicit expressions of the partial generating functions for \(|z| \leq 1\) are given as follows:

\[
P_0(z) = \left( \frac{1 - \rho}{1 + \eta} \right) \left[ \frac{\eta}{e^{\eta(z-1)}} \left( \frac{1 - \rho}{1 - \rho z} \right)^\frac{\eta}{\gamma} \right]
\]

\[
P_1(z) = \left( \frac{\eta + \rho}{1 - \rho z} \right) P_0(z)
\]

Where

\[\rho = \frac{\theta}{\mu}, \quad \eta = \frac{\lambda}{\mu}, \quad C = \lambda + \theta\]

and

\[
\pi_{0,0} = \left( \frac{1 - \rho}{1 + \eta} \right) e^{-\frac{\theta}{\gamma}(1 - \rho)\frac{\eta}{\gamma}}
\]

**Proof.** Substituting (8) in (5), we obtain

\[
(\lambda + \theta)P_0(z) + \theta z(P_0(z) + P_1(z)) = \mu P_1(z) + \theta zP_0(z)
\]

Rearranging the above result we get

\[
P_1(z) = \left( \frac{\lambda + \theta}{\mu - \theta z} \right) P_0(z)
\]

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Substituting (11) in (8)

\[ P'_0(z) = \frac{\theta}{\gamma} \left[ 1 + \frac{\lambda + \theta}{\mu - \theta z} \right] P_0(z) \]

We obtain the following differential equation

\[ \frac{P'_0(z)}{P_0(z)} = \frac{\theta}{\gamma} \left[ 1 + \frac{\lambda + \theta}{\mu - \theta z} \right] \]

Solving the above equation we get

\[ P_0(z) = P_0(1) \left[ e^{\frac{\theta}{\gamma}(z-1)} \left( \frac{1 - \rho}{1 - \rho z} \right) \right] \]

From (11), we get

\[ P_1(1) = \left( \frac{\lambda + \theta}{\mu - \theta} \right) P_0(1) \]

Consider the following normalizing condition

\[ P_0(1) + P_1(1) = 1 \quad (12) \]

We obtain

\[ P_0(1) = \frac{1 - \rho}{1 + \eta} \quad (13) \]

\[ P_1(1) = \frac{\eta + \rho}{1 + \eta} \quad (14) \]

**Corollary 2.** The equation (13) implies that the necessary and sufficient condition for the stability of the system is given by \( \rho < 1 \).

### 4.2 Explicit Expressions

**Theorem 3.** The following are the explicit expressions for the stationary distribution

\[ \pi_{0,j} = \pi_{0,0} \sum_{k=0}^{j} \frac{\theta^k}{\gamma^k j!} \left( \frac{C}{\gamma} \right)_{j-k} \frac{\rho^{j-k}}{(j-k)!} \quad (15) \]

\[ \pi_{1,j} = (\eta + \rho) \sum_{k=0}^{j} \pi_{0,k} \rho^{j-k} \quad (16) \]

for \( j \in \mathbb{Z}_+ \).
4.3 Recursive Formulae

**Theorem 4.** We compute the stationary probabilities using the following recursive formulae

\[
\pi_{0,j} = \frac{\theta [\pi_{0,j-1} + \pi_{1,j-1}]}{j\gamma} \quad j \in Z_+
\]

\[
\pi_{1,j} = \frac{[(\lambda + \theta)\pi_{0,j} + \theta\pi_{1,j-1}]}{\mu} \quad j \in Z_+
\]

\(\pi_{0,0}\) is given in theorem 1.

5 Stochastic Decomposition

In this section, we consider the stochastic decomposition property for the number of incoming susceptible population in the system (both orbit and server). We show that the number of incoming individuals can be decomposed into two independent random variables.

**Theorem 5.** In steady state, we have

\[L(t) = L_C(t) + L_M(t)\]

where \(L_C(t)\) denotes the number of incoming susceptible population in the system at time \(t\) and \(L_M(t)\) denote the number of incoming susceptible population when the infective is idle at time \(t\).

6 Conclusion

In this paper, we have constructed an SIS epidemic model using Markovian single server retrial queues. By means of the generating function approach, we have shown the stochastic decomposition property for the number of incoming susceptible individuals in the system. We have obtained a stability condition for the model. We have derived explicit formulae and recursive formulae for the joint stationary distribution of the number of repeated attempts and the state of the infective.

References


