Existence of Extremal Solution for Integral Boundary Value Problem of Non Linear Fractional Differential Equations

1R. Prahalatha and 2M.M. Shanmugapriya
1Department of Mathematics, Karpagam University,
Karpagam Academy of Higher Education.
prahalathav@gmail.com
2Department of Mathematics, Karpagam University,
Karpagam Academy of Higher Education.
priya.mirdu@gmail.com

Abstract

This paper investigates an explicit algorithm and explains the presence of maximal solutions for nonlinear fractional differential equations with integral and boundary conditions. An example is solved for the main concept of the study.
1. Introduction

Consider the following integral boundary value problem of nonlinear fractional differential equation

\[
\begin{align*}
{}^{C}D^{\alpha}v(t_i) &= g(t_i, v(t_i), v(\varphi(t_i))), \quad m < a < m+1, \quad m \geq 2, \quad m \in \mathbb{N}, \quad t_i \in I = [0,1] \\
v'(0) &= v'(0) = v''(0) = \ldots = v^{(m)}(0) = 0 \\
v(0) &= \int_{0}^{1} f(s, v(s))ds + l
\end{align*}
\]

(1.1)

where \( t \in I = [0,1] \) and \( {}^{C}D^{\alpha} \) is the standard Caputo fractional derivative and \( g \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \ f \in C(I \times \mathbb{R}, \mathbb{R}), \ \varphi \in C(I, I), \ l \geq 0. \)

Fractional differential equations are highly accepted powerful tools for solving many mathematical and scientific problems of real life situation. Fractional calculus has been examined carefully in various angles by many authors. The application and recent developments conceal the theoretical problems, Tension-Deformation, all kind of electrical networks, biomathematics, traffic control, viscoelasticity, weather forecasting, aerodynamics, economics, porous media, nanotechnology, modelling for physical phenomena showing anomalous diffusion, and so on. The main special feature of the differential operator with fractional order is its global phenomena that carry parental essence of several materials and real life movements. This view of operator with fractional order has stimulated the modellers to mould value of fractional calculus mechanism in converting real life problems to mathematical model for understanding basic concepts; the reader should refer the contents [1, 2].

Fractional integral boundary value problems make up a well framed and very interesting class of problems to solve. Fractional calculus with the boundary value problems have great attention in the last decade and plenty of results regarding the existence of solutions, based on several analytic techniques, can be understood in the literature survey [3-11]. The existence of solutions for fractional boundary value problems furnishes the basic investigation of the subject. It is primary to catch a problem solving form of solution for the given problem after proving the existence of a solution for that. The analytic strategy ensures the presence of solutions for the given problems and also accommodates meaningful and interesting factors of finding them. Also the monotone iterative technique coupled with the concept of upper and lower solutions have been used to prove the main results. For details see [12-20]. The above mentioned approach helps to explore the Extremal solutions for the problem (1.1).
The contents of this paper are ordered as given below.

In second section, a preliminary lemma has been proved to play a key role in proving the main concept. In third section, the main result is proved and an example is solved to make clear idea about the concept.

2. Preliminaries

Definition 2.1. $v(t_i)$ is called a lower solution of the problem (1.1), if

\[
\begin{align*}
\mathcal{D}^\alpha v(t_i) & \leq g(t_i, v(t_i), v(\varphi(t_i))), \quad m < a < m + 1, \ m \geq 2, \ m \in \mathbb{N}, \ t_i \in I = [0,1] \\
v'(0) &= v''(0) = v'''(0) = \ldots = v^{(m)}(0) = 0 \\
v(0) &\leq \frac{1}{0} \int f(s, v(s)) ds + l
\end{align*}
\]

Definition 2.2. $v(t_i)$ is called an upper solution of the problem (1.1), if

\[
\begin{align*}
\mathcal{D}^\alpha v(t_i) & \geq g(t_i, v(t_i), v(\varphi(t_i))), \quad m < a < m + 1, \ m \geq 2, \ m \in \mathbb{N}, \ t_i \in I = [0,1] \\
v'(0) &= v''(0) = v'''(0) = \ldots = v^{(m)}(0) = 0 \\
v(0) &\geq \frac{1}{0} \int f(s, v(s)) ds + l
\end{align*}
\]

Definition 2.3. The Riemann-Liouville fractional integral of order $\alpha$ for a function $g$ is defined as

\[
I^\alpha g(t_i) = \frac{1}{\Gamma(\alpha)} \left( t_i - s \right)^{\alpha-1} g(s) ds, \ a > 0
\]

provided that such integral exists.

Definition 2.4. The Caputo derivative of fractional order $\alpha$ for a differentiable function which is at least $m$ times absolutely continuous is defined as

\[
\mathcal{D}^\alpha g(t_i) = \frac{1}{\Gamma(m-a)} \left( t_i - s \right)^{m-a-1} g^{(m)}(s) ds, \ m = [\alpha] + 1
\]

where $[\alpha]$ denotes the integer part of the real number $\alpha$.

Lemma 2.1: Let $m < a \leq m + 1, \ m \geq 2, \ m \in \mathbb{N}, \ \alpha \neq 1$ and $x \in C[0,1]$. Then the linear fractional differential equations with integral boundary condition

\[
\begin{align*}
\mathcal{D}^\alpha v(t_i) &= x(t_i), \ \ 0 < t_i < 1 \\
v'(0) &= v''(0) = v'''(0) = \ldots = v^{(m)}(0) = 0 \\
v(0) &= \frac{1}{0} \int f(s, v(s)) ds + l
\end{align*}
\]
has a unique solution.

\[ v(t_i) = \int_0^1 G(t_i, s)x(s)ds + \frac{l}{1-\alpha} \]  

(2.2)

where

\[ G(t_i, s) = \begin{cases} \frac{a(t_i - s)^{\alpha-1}(1-\alpha) + \alpha(1-s)^{\alpha}}{(1-\alpha)\Gamma(a+1)} & 0 \leq s \leq t_i \leq 1 \\ \frac{\alpha(1-s)^{\alpha}}{(1-\alpha)\Gamma(a+1)} & 0 \leq t_i \leq s \leq 1 \end{cases} \]  

(2.3)

**Proof.** Transform the equation \(^cD^\alpha v(t_i) = x(t_i)\) to an equivalent integral equation.

\[ v(t_i) = I^\alpha x(t_i) + \sum_{j=0}^m p_j f_j = \frac{1}{\Gamma(a)} \int_0^1 (t_i - s)^{a-1} x(s)ds + \sum_{j=0}^m p_j f_j \]

for some \(p_j \in \mathbb{R}\) \((j = 0, 1, 2, \ldots, m)\).

Using the conditions \(v'(0) = v''(0) = \ldots = v^{(s)}(0) = 0\) and \(v(0) = \alpha \int_0^1 v(s)ds + l\), the result is \(p_1 = p_2 = p_3 = \ldots = p_n = 0\) and \(p_0 = \alpha \int_0^1 v(s)ds + l\).

Hence, it holds that

\[ v(t_i) = \frac{1}{\Gamma(a)} \int_0^1 (t_i - s)^{a-1} x(s)ds + \alpha \int_0^1 v(s)ds + l \]  

(2.4)

Let \(\int_0^1 v(s)ds = A\) and integrating both sides of (2.4), the following is obtained.

\[ A = \frac{1}{\Gamma(a)} \int_0^1 (1-s)^a x(s)ds + \alpha A + l = \frac{1}{\alpha} \int_0^1 (1-s)^a x(s)ds + \alpha A + l \]  

(2.5)

which gives \(A = \frac{1}{1-\alpha} \int_0^1 (1-s)^a x(s)ds + \frac{l}{1-\alpha}\).

The unique solution of the problem (2.1) is obtained by substituting the value of \(A\) in (2.4).

\[ v(t_i) = \frac{1}{\Gamma(a)} \int_0^1 (t_i - s)^{a-1} x(s)ds + \frac{\alpha}{1-\alpha} \int_0^1 t_i(1-s)^a x(s)ds + \frac{l\alpha}{1-\alpha} + l \]

\[ = \frac{\alpha(t_i - s)^{\alpha-1}(1-\alpha) + \alpha(1-s)^{\alpha}}{(1-\alpha)\Gamma(a+1)} x(s)ds + \frac{1}{(1-\alpha)\Gamma(a+1)} \int_0^1 \alpha(1-s)^a x(s)ds + \frac{l}{1-\alpha} \]

\[ v(t_i) = \int_0^1 G(t_i, s)x(s)ds + \frac{l}{1-\alpha}. \]
Hence the proof.

**Lemma 2.2.** If $0 \leq \alpha \leq 1$ and $\nu(t_i)$ satisfies

\[
\begin{align*}
\mathcal{C}D^\alpha \nu(t_i) &\geq 0, \ 0 < t_i < 1 \\
\nu'(0) = \nu''(0) = \nu'''(0) = \cdots = \nu^{(\alpha)}(0) = 0 \quad (2.6) \\
\nu(0) &\geq \alpha \int_0^1 \nu(s) ds
\end{align*}
\]

Then $\nu(t_i) \geq 0, \ \forall \ t_i \in I$.

**Proof.** It is known that the problem (2.1) has a unique solution which is proved by lemma 2.1

\[
\nu(t_i) = \frac{1}{1-\alpha} \int_0^1 G(t_i, s)x(s) ds + \frac{l}{1-\alpha}.
\]

Also satisfies Green’s function $G(t_i, s) \geq 0, \ t_i, s \in [0,1]$. Let $x(t_i) \geq 0$ and $l \geq 0$. From the above concept, the following is obtained.

\[
\begin{align*}
\mathcal{C}D^\alpha \nu(t_i) &\geq 0, \ 0 < t_i < 1 \\
\nu'(0) = \nu''(0) = \nu'''(0) = \cdots = \nu^{(\alpha)}(0) = 0 \quad (2.6) \\
\nu(0) &\geq \alpha \int_0^1 \nu(s) ds
\end{align*}
\]

Hence the proof.

### 3. Main Result

The following fixed point theorem is necessary to prove the main result.

**Theorem 3.1.** [21] Let $[c, d]$ be an ordered interval in a subset $X$ of an ordered Banach space $Y$ and let $P: [c, d] \rightarrow [c, d]$ be an increasing mapping. If for each convergence sequence $\{P_n\} \subset P([c, d])$, where $\{y_n\}$ is a monotone sequence in $[c, d]$, then the sequence of $P$ iteration of $c$ converges to the least fixed point $y$ of $P$ and the sequence of $P$ iteration of $d$ converges to the greatest fixed point $y^*$ of $P$. Moreover

\[
y_* = \min \{x \in [c, d]: x \geq P_x\} \quad \text{and} \quad y^* = \max \{x \in [c, d]: x \leq P_x\}
\]
Corollary 3.1. The following sequences \( \{v_m\}, \{w_m\} \) are computed by choosing initial values \( v_0(t_i), w_0(t_i) \).

\[
\begin{align*}
v_{m+1}(t_i) &= \int_0^t G(t_i, s) g\left(s, v_n(s), v_n(\phi(s))\right) ds + \frac{1}{1-\alpha} \int_0^t \left[f\left(s, v_n(s) - \alpha v_n(s)\right)\right] ds + l \\
w_{m+1}(t_i) &= \int_0^t G(t_i, s) g\left(s, w_n(s), w_n(\phi(s))\right) ds + \frac{1}{1-\alpha} \int_0^t \left[f\left(s, w_n(s) - \alpha w_n(s)\right)\right] ds + l
\end{align*}
\] (3.1)

Theorem 3.2. Consider \( v_0, w_0 \in BC(I, \mathbb{R}) \) as lower and upper solutions of (1.1) respectively such that \( v_0(t_i) \leq w_0(t_i), \forall t_i \in I \). Assume that the following conditions hold.

(A1) The function \( g \in C(I \times \mathbb{R}^2, \mathbb{R}) \) satisfies

\[
g(t, v, w) \geq g(t, v, w) \text{ for } v_0(t_i) \leq v \leq w_0(t_i). \quad v_0(\phi(t_i)) \leq w \leq w_0(\phi(t_i)) \quad \forall t_i \in I.
\]

(A2) There exists a constant \( 0 \leq \alpha \leq 1 \), such that

\[
f(t, v) - f(t, w) \geq \alpha (v - w) \text{ for } v_0(t_i) \leq v \leq w_0(t_i), \forall t_i \in I.
\]

Then there exists Extremal solutions \( v_*, w* \in [v_0, w_0] \) for the nonlinear integral boundary value problem (1.1). Also the Extremal solution obtained by using iterative sequences \( \{v_m\}, \{w_m\} \) which is formulated by corollary 3.1 converges to \( v_*, w^* \) respectively. Also

\[
v_0 \leq v_1 \leq v_2 \leq \ldots \leq v_m \leq \ldots \leq v_* \leq w^* \leq \ldots \leq w_m \leq \ldots \leq w_0.
\]

Proof. Consider the integral boundary value problem for linear fractional differential equation as follows

\[
\begin{align*}
^cD^\alpha v(t_i) &= g(t, h(t_i), h(\phi(t_i))), \quad 0 < t_i < 1 \\
v'(0) &= v'(0) = v''(0) = \ldots = v'^m(0) = 0 \\
v(0) &= \int_0^t [f(s, h(s)) + \alpha (v - h(s))] ds + l
\end{align*}
\] (3.2)

The above linear fractional differential problem has a unique solution which is derived using lemma 2.1. That is

\[
v(t_i) = \int_0^t G(t_i, s) g\left(s, h(s), h(\phi(s))\right) ds + \frac{1}{1-\alpha} \int_0^t \left[f\left(s, h(s) - \alpha h(s)\right)\right] ds + l
\]

where

\[
G(t_i, s) = \begin{cases} \frac{a(t_i-s)^{\alpha-1}(1-\alpha) + \alpha (1-s)^\alpha}{(1-\alpha)\Gamma(a+1)} & 0 \leq s \leq t_i \leq 1 \\
\frac{\alpha (1-s)^\alpha}{(1-\alpha)\Gamma(a+1)} & 0 \leq t_i \leq s \leq 1
\end{cases}
\]
for any \( h \in [v_0, w_0] \), define an operator \( A \) with \( v(t) = Ah(t) \). 

To show that: \([v_0, w_0]\) is a lower solution of the problem (1.1) and using (3.3), the following is obtained

\[
^{C}D^{\alpha}u(t_1) \geq 0, \quad 0 < t_1 < 1 \\
u'(0) = u^*(0) = u^{**}(0) = \ldots = u^{(m)}(0) = 0 \\
u(0) = \int_0^1 \left( f(s, v(s)) + \alpha(v - v_0)(s) \right) ds + l
\]

and

\[
^{C}D^{\alpha}w(t_1) = g(t, w_0(t_1), w_0(\varphi(t_1))), \quad 0 < t_1 < 1 \\
w_1'(0) = w_1^*(0) = w_1^{**}(0) = \ldots = w_1^{(m)}(0) = 0 \\
w_1(0) = \int_0^1 \left( f(s, w(s)) + \alpha(w - w_0)(s) \right) ds + l
\]

Set \( u = v_1 - v_0 \).

Since \( v_0 \) is a lower solution of the problem (1.1) and using (3.3), the following is obtained

\[
^{C}D^{\alpha}u(t_1) \geq 0, \quad 0 < t_1 < 1 \\
u'(0) = u^*(0) = u^{**}(0) = \ldots = u^{(m)}(0) = 0 \\
u(0) \geq \int_0^1 u(s) ds
\]

By lemma 2.2, \( u(t_1) \geq 0, \quad \forall t_1 \in I \). Hence \( Av_0 \geq v_0 \).

Also using (3.4) and upper solution definition, the result is \( Aw_0 \leq w_0 \).

Now let \( q = w_1 - v_1 \).

Applying the conditions (A1), (A2), (3.3) and (3.4), so the result is

\[
^{C}D^{\alpha}q(t_1) = g(t, w_0(t_1), w_0(\varphi(t_1))) - g(t, v_0(t_1), v_0(\varphi(t_1))) \geq 0 \\
q'(0) = q^*(0) = q^{**}(0) = \ldots = q^{(m)}(0) = 0 \\
q(0) = \int_0^1 \left( f(s, w(s)) + \alpha(w - w_0)(s) \right) ds + \int_0^1 \left( f(s, v(s)) + \alpha(v - v_0)(s) \right) ds \geq \int_0^1 q(s) ds
\]

Also by lemma 2.2, the result is \( q(t_1) \geq 0 \).

Thus \( Av_0 \leq Aw_0 \), \( v_0 \leq Av_0 \) and \( Aw_0 \leq w_0 \).

Hence \( A \) is non-decreasing and \( A : [v_0, w_0] \to [v_0, w_0] \).
Let \( \{u_m\} \) be a monotone iterative sequence in \([v_0, w_0]\).

Then \( v_0 \leq A u_m \leq w_0 \).

By Arzela-Ascoli theorem, the sequence \( \{A u_m\} \subset A([v_0, w_0]) \) converges.

Hence the convergence of the sequence of \( A \) iteration of \( v_0 \) to the least fixed point \( v^* \) is deduced with the help of theorem 3.1.

From the corollary 3.1, using the iterative sequences \( \{v_m\}, \{w_m\} \), the Extremal solutions are obtained.

This implies that the given problem (1.1) has Extremal solutions \( v^*, w^* \in [v_0, w_0] \).

Moreover, we have

\[
v_0 \leq v_1 \leq v_2 \leq \ldots \leq v_m \leq \ldots \leq v_0 \leq w_0 \leq \ldots \leq w_1 \leq w_0.
\]

Hence the proof.

**Example 3.1.** Consider a fractional boundary value problem with integral boundary conditions

\[
{}^cD^{\frac{5}{2}}y(t_c) = \frac{2}{5}t_i^{-\frac{1}{2}}v^2(t_i) + \frac{1}{5}e^{\alpha t_i^2}, \quad 0 < t_i < 1
\]

\[
v'(0) = v^*(0) = 0, \quad v(0) = \int_0^{\frac{1}{10}} e^{\alpha s^2} + \frac{1}{3} ds
\]

(3.7)

Take \( v_0 = 0, \ w_0 = 1 + t_i^2 \).

where \( v_0 \) and \( w_0 \) are lower and upper solutions of problem (3.7) respectively.

where \( g(t_i, v) = \frac{2}{5}t_i^{-\frac{1}{2}}v^2 + \frac{1}{5}e^{\alpha t_i^2} \) satisfies (42) with \( \alpha = \frac{1}{10} \).

Using corollary 3.1 and iterative sequences \( \{v_m\}, \{w_m\} \), hence the Extremal solutions are \( v^* \) and \( w^* \).

By theorem 3.2, the problem (3.1) has Extremal solutions \( v^*, w^* \in [0.2 + t_i^2 \frac{5}{2}] \).

### 4. Conclusion

An explicit algorithm has been investigated. The outmost solution for taken nonlinear fractional differential equations with prescribed conditions has been derived in clear and relevant example.
References


