STRONG RAINBOW VERTEX-COLORING OF CUBIC HALIN GRAPHS

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Abstract

A path $P$ in a non trivial connected graph $G$ is a rainbow path if no two internal vertices of $P$ have same color. A graph $G$ is strongly rainbow vertex-connected if every two vertices of $G$ are connected by at least one shortest rainbow path. The strong rainbow vertex-connection number, denoted by $srvc(G)$, is the minimum number of colors required to make a strongly rainbow vertex-connected graph $G$. In this paper we explore the strong rainbow vertex-connection number of complete cubic Halin graph and necklace graph.

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Key Words and Phrases: strong rainbow vertex-connection number, necklace graph, complete cubic Halin graph.

1 Introduction

Rainbow coloring has received much attention recently in the field of interconnection networks. The theory of rainbow coloring was
put forward by Chartrand in 2008 [2]. In [3], Krivelevich and Yuster proposed the concept of rainbow vertex-connection. The decisional version of rainbow connection number of an arbitrary graph is NP-Hard [1]. Let $G = (V, E)$ be a nontrivial connected graph. A vertex-coloring of a graph $G$ is a function from its vertex set to the set of natural numbers. A path $P$ in a vertex-colored graph $G$ is called a rainbow path if no two internal vertices get the same color. A vertex-colored graph $G$ is rainbow vertex-connected if every pair of vertices is connected by at least one rainbow path. Such a coloring is called a rainbow vertex-coloring. The rainbow vertex-connection number $rvc(G)$ is the minimum number of colors needed for rainbow vertex coloring of a graph $G$. We have $rvc(G) \leq n - 2$ (except for the singleton graph), and $rvc(G) = 0$ if and only if $G$ is a clique.

Similarly, a vertex-colored graph $G$ is strongly rainbow vertex-connected, if every pair of distinct vertices are connected by at least one shortest rainbow path. The minimum number of colors required to strongly rainbow color a graph $G$ is called the strong rainbow vertex-connection number, denoted by $srvc(G)$. If $G$ is a connected graph of order $n$, then $diam(G) - 1 \leq rvc(G) \leq srvc(G)$. Furthermore, $srvc(G) = 0$ if and only if $G$ is a complete graph and $srvc(G) = 1$ if and only if $diam(G) = 2$ [4]. Let $c(v)$ denote the color of the vertex $v \in V$. The distance between two vertices $u$ and $v$ in $G$, denoted by $d(u, v)$ is the length of a shortest path between them in $G$. The eccentricity $e(v)$ of a vertex $v$ in a connected graph $G$ is $\max d(u, v)$, for all $u \in V$. The maximum eccentricity of all vertices in a graph $G$ is called the diameter $diam(G)$ of the graph $G$. This paper focuses on the strong rainbow vertex-connection number of the cubic planar graphs namely complete cubic Halin graph and necklace graph.

2 Strong rainbow vertex connection in complete cubic Halin graph

In this section, we investigate the strong rainbow vertex-connection in complete cubic Halin graph which has a very large strong rainbow vertex-connection number.

A nontrivial connected graph $G$ is called a tree if it contains no cycles. A leaf in $G$ is a vertex of degree 1. A Halin graph $G$ is
constructed from a tree $T$ with at least four vertices by connecting all the leaves of $T$ with a cycle $C$ which passes through the boundary of $T$. The tree $T$ and the cycle $C$ are called the characteristic tree and the adjoint cycle of $G$ respectively [6]. In a cubic Halin graph, the characteristic tree $T$ is a caterpillar and the degree of each vertex of $T$ is 3.

A cubic tree is a tree such that every internal vertex has degree 3. For $n \geq 0$, a complete cubic tree $T_n$ is a cubic tree of height $n+1$ with a root vertex $v_0$ such that all its leaves are at the same distance $n+1$ from $v_0$ [5]. The level of a vertex in $T_n$ is defined to be the distance from the root vertex to that vertex. For any edge $e = uv$ of $T_n$, assume $v$ is a child of $u$. The level of $e$ is defined to be the level of $v$. The root vertex is situated at level 0. The complete cubic tree $T_n$ has $n+1$ levels and all the internal vertices between level 1 and level $n$ of $T_n$ have two children. A complete cubic Halin graph $H_n$ is a cubic Halin graph whose characteristic tree is $T_n$. The chromatic number of $H_n$ is 3. The diameter of $H$ is $2(n+1)$. Figure 1 is an example for complete cubic Halin graph of dimension 2. Let $v_0$ be the root vertex of the complete cubic tree $T_n$ at level 0. Vertices at level $j$, $1 \leq j \leq n$ are of the form \{ $v^j_i$ | $1 \leq i \leq 3 \cdot 2^{j-1}$ \}.

\begin{center}
\includegraphics[width=0.5\textwidth]{H2.png}
\end{center}

Figure 1: $H_2$

**Proposition 1.** *The strong rainbow vertex-connection numbers* of the complete cubic Halin graphs $H_1$, $H_2$ and $H_3$ are

(i) $srvc(H_1) = 2$

(ii) $srvc(H_2) = 6$

(iii) $srvc(H_3) = 13$
Proof. (i) The complete cubic Halin graph $H_1$ consists of a complete cubic tree of height 2 and the adjoint cycle of length 6. The diameter of $H_1$ is 3. This implies that the two internal vertices of a path of length 3 should be distinct in color. The root vertex is assigned with a color 1. The level 1 vertices are assigned with a color 2. The cycle vertices can be assigned with the colors 1 and 2 alternately. It is easy to check that any two vertices are connected by at least one shortest rainbow path.

(ii) The diameter of $H_2$ is 6. The root vertex is assigned with a color 1. Level 1 vertices are assigned with colors 2, 3, 4. We note that there exists no shortest path containing three consecutive level 2 vertices in it. That is, every three consecutive vertices in level 2 can be assigned with the same color subject to the condition that the color of the first three vertices should be distinct from the next three vertices. Thus, level 2 vertices are assigned with colors 5, 5, 5, 6, 6, 6 starting at any vertex cyclically. We find that the cycle vertices can be assigned with already used colors. Therefore the cycle vertices are assigned with the colors 1, 2, 3, 4 cyclically starting at any vertex. The above coloring scheme gives an optimum coloring of $H_2$ using 6 colors.

(iii) The complete cubic Halin graph $H_3$ consists of a complete cubic tree $T_3$ of height 4 and the cycle of length 24. The root vertex, level 1 vertices and level 2 vertices require distinct colors for rainbow vertex coloring. The level 3 vertices in $T_3$ are $\{v_i^3 \mid 1 \leq i \leq 12\}$. Four consecutive level 3 vertices receive the same color if the first two vertices and the next two vertices do not have common ancestor at level 1. Consequently 3 colors are required for level 3 vertices.

Figure 2: $src(H_2) = 6$
Thus totally 13 colors are required for $T_3$. The cycle vertices can be assigned with the same colors that are used for $T_3$. It is easy to see that every two vertices are connected by at least one shortest rainbow path.

**Theorem 2.** For $n \geq 4$, the strong rainbow vertex-connection number of $H_n$ is $srvc(H_n) = 3(2^n - 1 + (3.2^{n-4})) + 1$.

**Proof.** Let $v_0$ be the root vertex of the complete cubic tree $T_n$. Let $S_{v_1}, S_{v_2}, S_{v_3}$ be the sub trees of the complete cubic tree $T_n$ obtained by deleting the root vertex $v_0$. The level 1 vertices $v_1^1$, $v_2^1$ and $v_3^1$ are the roots of the sub trees $S_{v_1}, S_{v_2}$ and $S_{v_3}$ respectively. Clearly, $T_n = S_{v_1} \cup S_{v_2} \cup S_{v_3} \cup v_0$ and $S_{v_i} \cap S_{v_j} = \phi$ for $i \neq j$. See fig. 3. We note that $|V(S_{v_i})| = 2^{n+1} - 1$ and $|E(S_{v_i})| = 2^{n+1} - 2$, $1 \leq i \leq 3$.

![Figure 3: $S_{v_1}, S_{v_2}, S_{v_3}$ are the subtrees of the complete cubic tree $T_n$](image)

In $T_n$, the vertices at level $j$, $1 \leq j \leq n$ are of the form $\{v_i^j \mid 1 \leq i \leq 3.2^{j-1}\}$. There are $3(2^n - 1)$ vertices between level 1 and level $n$. We note that there exists unique shortest path between the root vertex $v_0$ and the vertices at level $n$. The leaf edge of the complete cubic tree $T_n$ is incident at a vertex of the cycle $C_m$ where $m = 3.2^n$. The level $n+1$ vertices of $T_n$ are the vertices of the cycle $C_m$. Label the vertices of $C_m$ as $u_1, u_2, \ldots, u_{3.2^n}$ in the clockwise
sense. To achieve a strong rainbow vertex coloring, we proceed with
the following algorithm. In step 1, the root vertex is assigned with
a distinct color. In Step 2, all the internal vertices between level 1
and level \( n - 1 \) are assigned with distinct colors. Step 3 focuses on
finding the minimum number of consecutive vertices in level \( n \) that
can take the same color so as to preserve rainbow connectivity and
assigning colors to level \( n \) vertices. Step 4 assigns colors to all the
cycle vertices.

Coloring Algorithm

(i) The root vertex \( v_0 \) is assigned with a distinct color.
(ii) The vertices from level 1 to level \( n - 1 \) can be assigned with
distinct colors. Since there exists a unique shortest path from the
root vertex \( v_0 \) to level \( n - 1 \) vertices, they should be assigned with
distinct colors. We find that there are \( 3(2^{n-1} - 1) \) vertices from
level 1 to level \( n - 1 \). They should be assigned with \( 3(2^{n-1} - 1) \)
distinct colors.
(iii) The level \( n \) vertices of \( T_n \) can be colored as follows. The level \( n \)
vertices of \( T_n \) in \( S_{v_1}^1, S_{v_2}^1 \) and \( S_{v_3}^1 \) are labeled as \( \{v_1^n, v_1^{n+1}, \ldots, v_2^{n-1}\} \),
\( \{v_2^{n-1+1}, v_2^{n-1+2}, \ldots, v_2^n\} \) and \( \{v_3^n, v_3^{n+2}, \ldots, v_3^{n+2n-1}\} \) respectively.
Consider the sub tree \( S_{v_i}^1 \). The vertices \( v_i^n \) and \( v_i^{n+1} \),
\( 1 \leq i \leq 2^{n-1} \) and \( i \equiv 1 \) (mod 2) are siblings as they have the
same parent vertex at level \( n - 1 \). We note that the vertices
\( v_i^n, v_{i+1}^n, v_{i+2}^n, v_{i+3}^n \) have a common ancestor at level \( n - 2 \) and
the vertices \( v_{i+4}^n, v_{i+5}^n, v_{i+6}^n, v_{i+7}^n \) have a common ancestor at
level \( n - 2 \) such that the common ancestor at (level \( n - 2 \) of
the vertices \( v_i^n, v_{i+1}^n, v_{i+2}^n, v_{i+3}^n \)) and the common ancestor at (level
\( n - 2 \) of the vertices \( v_{i+4}^n, v_{i+5}^n, v_{i+6}^n, v_{i+7}^n \)) are different vertices.
If so, then \( c(v_i^n) = c(v_{i+1}^n) \), \( c(v_{i+2}^n) = c(v_{i+3}^n) = c(v_{i+1}^n) =
c(v_{i+5}^n) \), \( c(v_{i+6}^n) = c(v_{i+7}^n) \) with \( c(v_{i+1}^n) \neq c(v_{i+2}^n) \) and \( c(v_{i+5}^n) \neq
c(v_{i+6}^n) \), \( 1 \leq i \leq 2^{n-1} \) and \( i \equiv 1 \) (mod 8). Color the vertices
\( \{v_1^n, v_2^n, \ldots, v_{2^n-1}^n\} \) in the sub tree \( S_{v_1}^1 \) using the coloring pattern
given above, we find that the level \( n \) vertices of \( T_n \) in \( S_{v_1}^1 \) require
\( 3.2^{n-4} \) distinct colors. Similarly applying the same coloring pattern
to the level \( n \) vertices \( \{v_{2^n-1+1}^n, v_{2^n-1+2}^n, \ldots, v_{2^n-1+4}^n\} \) in \( S_{v_1}^1 \) and
\( \{v_{2^n+1}^n, v_{2^n+2}^n, \ldots, v_{2^n+2n-1}^n\} \) in \( S_{v_3}^1 \) of \( T_n \), we find that the level \( n \)
vertices of \( T_n \) require \( 9.2^{n-4} \) distinct colors.
(iv) Level \( n + 1 \) vertices can be colored as follows: The level
\( n + 1 \) vertices are the vertices of the cycle \( C_m \) where \( m = 3.2^n \). To
color the vertices of the cycle, we make use of the colors already
assigned between the root vertex \( v_0 \) and the level \( n - 1 \) vertices. Since \( \text{diam}(H_n) = 2(n + 1) \), any path of length \( \text{diam}(H_n) \) will have \( 2n + 1 \) internal vertices in it. Also if \( P \) is a path of length \( 2n + 1 \) and the end vertices of \( P \) are level \( n + 1 \) vertices, then the shortest path connecting them must pass through the cycle vertices. In this case, the path \( P \) has \( 2n \) consecutive cycle vertices as its internal vertices. The path \( P \) will be a rainbow path if the \( 2n \) consecutive cycle vertices are assigned with distinct colors. This implies that along the cycle every \( 2n \) consecutive vertices should be distinct in color. Assign \( 2n \) distinct colors to the \( 2n \) consecutive vertices \( \lfloor \frac{3.2^n}{2n} \rfloor \) times cyclically. If \( 2n \) divides \( 3.2^n \), then all the \( 3.2^n \) vertices are assigned with \( 2n \) distinct colors cyclically. If not, there can be at most \( 2n - 1 \) remaining uncolored cycle vertices. They can be assigned with the colors distinct from the colors that are already assigned to the cycle vertices. That is, we can use the colors that are already assigned between the root vertex \( v_0 \) and the level \( n - 1 \) vertices. But they should be distinct from the colors that are already assigned to the cycle vertices. Thus the coloring of \( H_n \) given in the above algorithm is a strong rainbow vertex coloring as every two distinct vertices are connected by at least one shortest rainbow path. Thus \( 3(2^{n-1} - 1 + (3.2^{n-4})) + 1 \) colors are required for the complete cubic Halin graph \( H_n \). 

\[\square\]

Figure 4: Strong rainbow vertex-coloring of \( H_4 \)
Observation 3. \( srvc(H_n) \) is closer to \( \left\lfloor \frac{V(H_n)}{3} \right\rfloor \).

3 Strong rainbow vertex connection in necklace graph \( N_{\epsilon_n} \)

Suppose \( G \) is a Halin graph of order \( 2n+2 \) with a caterpillar \( T \) as its characteristic tree, \( n \geq 1 \). We denote the vertices along the spine \( P_n \) by \( v_1, v_2, \ldots, v_n \). The vertices adjacent with \( v_1 \) are denoted by \( v_0 \) and \( v'_1 \). The vertices adjacent with \( v_n \) are denoted by \( v_{n+1} \) and \( v'_n \). Other leaf adjacent with \( v_i \) is denoted by \( v'_i \), \( 2 \leq i \leq n - 1 \). Note that the vertices \( v_0, v'_1, \ldots, v_n, v_{n+1} \) lie on the adjoint cycle \( C_{n+2} \). Moreover, if the vertices \( v_0, v'_1, \ldots, v'_n, v_{n+1} \) in \( C_{n+2} \) are in order and lie on the same side of the spine, then \( G \) is called a necklace and is denoted by \( N_{\epsilon_n} \) \cite{5}. The diameter of \( N_{\epsilon_n} \) is \( \left\lceil \frac{n}{2} \right\rceil + 1 \). The chromatic number \( \chi(N_{\epsilon_n}) = 3 \).

![Figure 5: \( N_{\epsilon_3} \)](image)

Theorem 4. The strong rainbow vertex-connection number of necklace graph \( N_{\epsilon_n} \) is \( srvc(N_{\epsilon_n}) = \begin{cases} \text{diam}(N_{\epsilon_n}) - 1 & \text{for } 1 \leq n \leq 7 \\ \text{diam}(N_{\epsilon_n}) & \text{for } n \geq 8 \end{cases} \)

Proof. For \( n = 1 \), the graph is \( K_4 \). \( srvc(K_4) = 0 \). For \( n = 2 \), all the vertices are assigned with color 1. For \( n = 3 \), the vertices \( v_0, v_1, v_2, v_3, v_4 \) are assigned with the colors 1, 2, 1, 2, 1 respectively. Similarly the vertices \( v'_1, v'_2, v'_3 \) are assigned with the colors 2, 1, 2 respectively. This gives a strong rainbow vertex coloring using two colors. For \( n = 4 \), the vertices \( v_0, v_1, v_2, v_3, v_4, v_5 \) are assigned with the colors 1, 2, 1, 2, 1, 2 respectively and the vertices \( v'_1, v'_2, v'_3, v'_4 \) are assigned with the colors 2, 1, 2, 1 respectively. This gives a strong rainbow vertex coloring using two colors. For \( n = 5 \), an illustration
is given in fig. 6. For $n = 6$, the vertices $v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7$ are assigned with the colors $1, 2, 1, 2, 3, 1, 2$ respectively and the vertices $v'_1, v'_2, v'_3, v'_4, v'_5, v'_6$ are assigned with the colors $2, 1, 2, 3, 1, 2$ respectively. This gives a strong rainbow vertex coloring using three colors. Similarly we can find the strong rainbow vertex-connection number for $n = 7$ using 4 colors.

For $n \geq 8$, the coloring algorithm is as follows. $c(v_i) = i + 1$ for $0 \leq i \leq \lceil \frac{n}{2} \rceil$ and $c(v_i) = i - \lceil \frac{n}{2} \rceil$ for $\lceil \frac{n}{2} \rceil + 1 \leq i \leq n + 1$. Also $c(v_i) = c(v'_i)$ for $1 \leq i \leq n$. It is easy to check that every two vertices are connected by at least one shortest rainbow path.

![Figure 6: Strong rainbow vertex coloring of $Ne_5$](image)

Figure 6: Strong rainbow vertex coloring of $Ne_5$

\[ \square \]

4 Conclusion

In this paper, the strong rainbow vertex-connection number of complete cubic Halin graph and necklace graph have been computed. It is observed that $\text{srvc}(H_n)$ is closer to $\left\lfloor \frac{V(H_n)}{3} \right\rfloor$ and $\text{srvc}(Ne_n) = \text{diam}(Ne_n)$, $n \geq 8$. It would be interesting to study the strong rainbow vertex-connection numbers of variants of cubic Halin graphs.

References


