A profitable shipping method for solving capacitated transportation problems with bounds on rim conditions economically

P. Pandian ¹ and K.Kavitha ²*
¹,²Department of Mathematics, School of Advanced Sciences
VIT University, Vellore-14, Tamilnadu, India
¹pandian61@rediffmail.com and ²k.kavitha@vit.ac.in

February 10, 2017

Abstract

A capacitated transportation problem with bounds on the rim conditions is considered in this paper. A new method namely, profitable shipping method is proposed for solving the capacitated transportation problem with bounds on the rim conditions economically.

AMS Subject Classification: 90B06, 91B82, 97M40
Key Words and Phrases: Capacitated transportation problem, Optimal solution, Profitable shipping method.

1 Introduction

Capacitated transportation (CT) problem with bounds on rim (BOR) conditions is a special type of the problem in which routes transportation capacity, origins supply and destinations demand have

*Correspondence Author
lower and upper bounds. Many researchers like, Dahiya and Verma [3], Gupta and Arora [4], Jain and Arya [8], proposed various methods for solving CT problems with or without additional conditions. The minimization of dead-mileage assessed in terms of running buses from various depots to starting points was studied by Charnes and Klingman [2]. The randomized CT problem was analyzed by Hassain and Zemel [7]. Dahiya and Verma [3] presented a method for solving the CT problem with BOR conditions and they also, studied the problem with restricted flow and analyzed it in the paradoxical situation. Gupta and Arora [5] proposed an algorithm for solving a capacitated linear plus linear fractional transportation problem (TP) with restricted flow.

In the TP, the unit shipping cost of a shipping plan provides the average cost spent in the shipping plan. So, the minimization of the unit shipping cost in the transportation problem is far better than the minimization of the total transportation cost.

2 Capacitated Transportation Problem with Bounds on Rim Conditions

The CT problem with BOR conditions states as follows: In a company, there are $m$ origins, which contain various amounts of a commodity that has to be shipped to $n$ destinations. Let the minimum and maximum number of shipping quantity from the origin $i$ to the destination $j$ be $L_{ij}$ and $U_{ij}$ respectively.

Minimize $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$ (P)

subject to, $a_i \leq \sum_{j=1}^{n} x_{ij} \leq A_i$, for $i \in I$;

$b_j \leq \sum_{i=1}^{m} x_{ij} \leq B_j$, for $j \in J$;

$L_{ij} \leq x_{ij} \leq U_{ij}$ for $i \in I$ and $j \in J$, $x_{ij}$ are integers (3)

where $I = \{1, 2, \cdots, m\}$ and $J = \{1, 2, \cdots, n\}$.

The unit shipping cost of the problem (P) for its feasible solution
\[ X = \{x_{ij}; i \in I, j \in J\}, S(X) \text{ is given as } S(X) = \frac{\left( \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \right)}{\left( \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \right)}. \]

The allotment \( \{x_{ij} = L_{ij}; i \in I, j \in J\} \) to the problem (P) is known as the decision variable base level allotment to the problem (P). Now, we construct a problem from the problem (P) called, above base level problem, \((P_1)\) as given below:

\[
\text{Minimiwe } W = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} y_{ij} \quad (P_1)
\]

subject to

\[
a'_i \leq \sum_{j=1}^{n} y_{ij} \leq A'_i, \text{ for } i \in I; b'_j \leq \sum_{i=1}^{m} y_{ij} \leq B'_j, \text{ for } j \in J;
\]

\[ y_{ij} \geq 0, y_{ij} \leq T_{ij} \text{ for all } i \in I, j \in I \text{ and integers} \]

where \( y_{ij} = x_{ij} - L_{ij} \); \( T_{ij} = U_{ij} - L_{ij} \)

\[
a'_i = \begin{cases} a_i - \sum_{j=1}^{n} L_{ij} : a_i > \sum_{j=1}^{n} L_{ij} & A'_i = \begin{cases} A_i - \sum_{j=1}^{n} L_{ij} : A_i > \sum_{j=1}^{n} L_{ij} \\ 0 : \text{otherwise} \end{cases} \\ 0 : \text{otherwise} \end{cases}
\]

\[
b'_j = \begin{cases} b_j - \sum_{i=1}^{m} L_{ij} : b_j > \sum_{i=1}^{m} L_{ij} & B'_j = \begin{cases} B_j - \sum_{i=1}^{m} L_{ij} : B_j > \sum_{i=1}^{m} L_{ij} \\ 0 : \text{otherwise} \end{cases} \\ 0 : \text{otherwise} \end{cases}
\]

Now, we construct a problem \((L)\) from the problem \((P_1)\) called, balanced least level supply demand problem as follows:

\[
\text{Minimiwe } Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} y_{ij} \quad (L)
\]

subject to \( \sum_{j=1}^{n} y_{ij} = \bar{a}_i, \text{ for } i \in I; \sum_{j=1}^{m} y_{ij} = \bar{b}_j, \text{ for } j \in J; \)

\[ y_{ij} \geq 0, y_{ij} \leq T_{ij} \text{ for all } i \in I, j \in J \text{ and integers} \]

where \( \bar{a}_i \in [a'_i, A'_i] \); \( \bar{b}_j \in [b'_j, B'_j] \) and

\[
\sum_{i=1}^{m} \bar{a}_i = \sum_{j=1}^{n} \bar{b}_j = \text{maximum of } \left\{ \sum_{i=1}^{m} a'_i, \sum_{j=1}^{n} b'_j \right\}.
\]
Now, the following theorem connects the feasible solution to the problem (P) and the optimal solution of the problem (L).

**Theorem 1.** If \( Y = \{ y_{ij}^0, i \in I, j \in J \} \) is an optimal solution of the problem (L) and the allotment \( \{ x_{ij} = L_{ij}; i \in I, j \in J \} \) is the decision variable base level allotment to the problem (P), then \( X^0 = \{ x_{ij}^0 = y_{ij}^0 + L_{ij}, i \in I, j \in J \} \) is a feasible solution of the problem (P).

**Proof.** Now, since \( Y = \{ y_{ij}^0, i \in I, j \in J \} \) is an optimal solution of the problem (L), \( Y \) is a feasible solution to the problem (P). Now, since \( Y = \{ y_{ij}^0, i \in I, j \in J \} \) is a feasible solution to the problem (P) and the allotment \( \{ x_{ij} = L_{ij}; i \in I, j \in J \} \) is the decision variable base level allotment to the problem (P), we can conclude that the allotment set \( X^0 = \{ x_{ij}^0 = y_{ij}^0 + L_{ij}, i \in I, j \in J \} \) is a feasible solution of problem (P). Hence, the theorem is proved.

Now, the USC problem for the problem (P) is given below:

\[
\text{Minimize } S = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \right) / \left( \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \right) \quad (U)
\]

subject to (1), (2) and (3) are satisfied.

**Theorem 2.** Let \( X^0 = \{ x_{ij}^0, i \in I, j \in J \} \) be a feasible solution to the problem (P) with the unit shipping cost \( S^0 \). If \( c^{n} \leq S^0 \), for some \( r \) and \( t \) then, a feasible solution to the problem (P) is the allotment set \( X^1 = \{ x_{ij}^0, i \in I, j \in J, i \neq r, j \neq t \} \cup \{ x_{rt}^0 + p_{rt}^0 \} \) with unit shipping cost \( S^1 \leq S^0 \) where

\[
p_{rt}^0 = \min \left\{ U_{rt} - x_{rt}^0, A_r \sum_{j=1}^{n} x_{ij}^0 B_t - \sum_{i=1}^{m} x_{ij}^0 \right\}. \quad (4)
\]

**Proof.** Now, since \( X^0 \) is a feasible solution to the problem (P) and from (4), a feasible solution to the problem (P) is the allotment set \( X^1 = \{ x_{ij}^0, i \in I, j \in J, i \neq r, j \neq t \} \cup \{ x_{rt}^0 + p_{rt}^0 \} \).

Now, \( S^0 = S(X^0) = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}^0 \right) / \left( \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^0 \right) \). This implies,

\[
S^0 \left( \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^0 \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}^0. \quad (5)
\]
Now, \( S^1 = S(X^1) = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}^0 + crtp^0_{rt} \right) / \left( \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^0 + p^0_{rt} \right) \)

\[
S^0 \left( \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^0}{\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^0 + p^0_{rt}} \right) \quad \text{(by (5))} = S^0 - \frac{(S^0 - c_{rt})p^0_{rt}}{\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^0 + p^0_{rt}} \leq S^0.
\]

Therefore, \( S^1 \leq S^0 \). Hence, the theorem is proved.

Now, the sufficient optimality theorem for a feasible solution to the problem (U) to be optimal has been established as follows:

**Theorem 3.** Let \( X^0 = \{x_{ij}^0, i \in I, j \in J\} \) be a feasible solution to the problem (P) with the unit shipping cost \( S^0 \). If each cell \((i, j)\) in the transportation table for the problem (P) satisfies at least one of the following conditions:

(i) The cost of the cell \((i, j)\), \( c_{ij} > S^0 \).

(ii) The cost of the cell \((i, j)\), \( c_{ij} > S^0 \) and its allotment should not be increased.

Then, \( X^0 \) is an optimal solution of problem (U).

**Proof.** Cell \((i, j)\) in the transportation table is with cost \( c_{ij} \).

**Case(i):** If \( c_{ij} > S^0 \), the cell \((i, j)\) cannot be used to construct an improved feasible solution to the problem (P) such that its unit shipping cost is less than \( S^0 \) by the Theorem 2.

**Case(ii):** If \( c_{ij} \leq S^0 \) and the allotment of the cell \((i, j)\) should not be increased, then a new feasible solution to the problem (U) cannot be obtained by increasing the assignment of the cell \((i, j)\) without affecting the cell capacity and the BOR conditions.

Therefore, from the Case (i) and the Case (ii) and by the Theorem 2., we can conclude that \( X^0 = \{x_{ij}^0, i \in I, j \in J\} \) is an optimal solution of the problem (U). Hence, the theorem is proved. The proposed method proceeds as follows:

**Step 1:** Obtain the decision variable base level allotment for the given problem (P). Let it be \( \{x_{ij}, i \in I, j \in J\} \).

**Step 2:** Construct the above base level problem \((P_1)\) from the given problem (P).

**Step 3:** Construct the balanced least level supply-demand problem
(L) from the problem \((P_1)\) obtained in the Step 2.

**Step 4:** Solve the problem \((L)\) obtained in the Step 3. using the transportation algorithm. Let it be \(\{y_{ij}^0, i \in I, j \in J\}\).

**Step 5:** To the given problem \((P)\), we obtain the feasible solution \(W = \{z_{ij}^0 = L_{ij} + y_{ij}^0, i \in I, j \in J\}\) by the Theorem 1.

**Step 6:** Modify the allotment in each allotment cell of the feasible solution to the problem \((P)\) obtained from the Step 5. starting from the maximum cost to minimum cost of the cell as follows:

Let \((\alpha, \beta)\) be a considering allotment cell. If the \(\alpha^{th}\) row and \(\beta^{th}\) column have allotment cells with less cost than the cost of the cell \((\alpha, \beta)\), transfer the maximum possible amount from the cell \((\alpha, \beta)\) to the allotment cells of the \(\alpha^{th}\) row other than the cell \((\alpha, \beta)\) and the allotment cells of the \(\beta^{th}\) column other than the cell \((\alpha, \beta)\) starting from the least cost cell to maximum cost cell with maximum possible allotment such that the cell capacity and the BOR conditions are satisfied.

**Step 7:** Compute the unit shipping cost for the modified feasible solution \(X^0\) obtained in the Step 6. Let it be \(S^0\).

**Step 8:** Determine a cell in transportation table such that its cost is less than \(S^0\) and its allotment will be increased. If such a cell does not exist, go to the Step 12.

**Step 9:** Find the least cost \(c_{rt}\), for some \(r\) and \(t\) such that \(c_{rt} \leq S^0\) and \(x_{rt}\) does not attain its maximum possible value.

**Step 10:** Construct a modified feasible solution \(X^1\) to the problem \((P)\) (by the Theorem 2.)

**Step 11:** Compute the unit shipping cost for the modified feasible solution \(X^1\). Let it be \(S^1\). Then, go to the Step 8.

**Step 12.** The feasible solution of the problem \((P)\) obtained from Step 8. is an optimal solution of the problem \((U)\) by Theorem 3. \(\square\)

**Example 4.**

Minimize \(Z = \sum_{i=1}^{3} \sum_{j=1}^{3} c_{ij}x_{ij}\) \hspace{1cm} (P)

subject to

\[
20 \leq \sum_{j=1}^{3} x_{1j} \leq 55; 30 \leq \sum_{j=1}^{3} x_{2j} \leq 70; 15 \leq \sum_{j=1}^{3} x_{3j} \leq 45;
\]
\[20 \leq \sum_{i=1}^{2} x_{i1} \leq 55; 10 \leq \sum_{i=1}^{2} x_{i2} \leq 50; 35 \leq \sum_{i=1}^{2} x_{i3} \leq 65;\]

\[5 \leq x_{11} \leq 25; 0 \leq x_{12} \leq 20; 0 \leq x_{13} \leq 10; 5 \leq x_{21} \leq 20;\]

\[2 \leq x_{22} \leq 15; 3 \leq x_{23} \leq 35; 0 \leq x_{31} \leq 10; 4 \leq x_{32} \leq 15; 1 \leq x_{33} \leq 20\]

and \( x_{ij} \in I = \{1, 2, 3\} \) and \( j \in J = \{1, 2, 3\} \) are integers \hspace{1cm} (6)

where the cost matrix, \([c_{ij}] = \begin{pmatrix} 5 & 9 & 9 \\ 4 & 6 & 2 \\ 2 & 1 & 1 \end{pmatrix}\)

Now, the USC problem for the above problem is given below:

\[(U) \quad \text{minimize} \quad S = \left( \sum_{i=1}^{3} \sum_{j=1}^{3} c_{ij} x_{ij} \right) / \left( \sum_{i=1}^{3} \sum_{j=1}^{3} x_{ij} \right) \quad \text{subject to} \]

(6) is satisfied.

Now, by the Step 1., the decision variable base level allotment of the given by \( x_{11} = 5; x_{21} = 5; x_{22} = 2; x_{23} = 3; x_{32} = 4 \) and \( x_{33} = 1 \).

By Step 2 and the Step 3., the balanced least level supply-demand problem of the above base level problem is:

\[0 \leq y_{11} \leq 20; 0 \leq y_{12} \leq 20; 0 \leq y_{13} \leq 10; 0 \leq y_{21} \leq 15; 0 \leq y_{22} \leq 13; 0 \leq y_{23} \leq 32; 0 \leq y_{31} \leq 10; 0 \leq y_{32} \leq 11; \] and \( 0 \leq y_{33} \leq 19 \).

Now, by the Step 4. to the Step 12., we obtain the following improved feasible solution to the given problem \((P)\) with unit shipping cost \(257/107 = 2.4019\).

Now, the above obtained results are summarized in Table 1, where \( TSC - \text{Total shipping cost}; \) \( TNTI - \text{Total number of transporting items} \) and \( USC - \text{Unit shipping cost} \).

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Shipping plan</th>
<th>TSC</th>
<th>TNTI</th>
<th>USC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x_{11} = 15; x_{13} = 5; x_{21} = 5; ) ( x_{22} = 2; x_{23} = 23; x_{32} = 8; x_{33} = 7 )</td>
<td>213</td>
<td>65</td>
<td>3.277</td>
</tr>
<tr>
<td>2</td>
<td>( x_{11} = 20; x_{21} = 5; x_{22} = 2; ) ( x_{23} = 23; x_{32} = 8; x_{33} = 12 )</td>
<td>198</td>
<td>70</td>
<td>2.829</td>
</tr>
<tr>
<td>3</td>
<td>( x_{11} = 20; x_{21} = 5; x_{22} = 2; ) ( x_{23} = 23; x_{23} = 15; x_{33} = 20 )</td>
<td>213</td>
<td>85</td>
<td>2.5059</td>
</tr>
<tr>
<td>4</td>
<td>( x_{11} = 20; x_{21} = 5; x_{22} = 2; x_{23} = 35; ) ( x_{31} = 10; x_{32} = 15; x_{33} = 20 )</td>
<td>257</td>
<td>107</td>
<td>2.4019</td>
</tr>
</tbody>
</table>
3 Conclusion

In this paper, we consider a CT problem with BOR conditions. The USC problem for the CT problem with BOR conditions is constructed. The proposed method is based on the USC problem for the given problem and provides a better feasible solution than the optimal solution to the CT problem with BOR conditions.

References


