ODD-EVEN GRACEFUL LABELING OF CYCLES AND CYCLE RELATED GRAPHS

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Abstract

The Odd-Even graceful labeling of a graph $G$ with $q$ edges means that there is an injection $f : V(G)$ to \{1, 3, 5, \ldots, 2q+1\} such that, when each edge $uv$ is assigned the label $|f(u) - f(v)|$, the resulting edge labels are \{2, 4, 6, \ldots, 2q\}. A graph which admits an odd-even graceful labeling is called an odd-even graceful graph. In this paper we will prove for $n \equiv 0, 3 \pmod{4}$ cycles $C_n$, hairy cycles $C_n \odot mK_1$, and one-point union of cycles of length 4 are odd-even graceful.

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1 Introduction

Labeling have a wide range of applications in different areas such as coding theory, X-ray crystallography, radar, astronomy, circuit design and communication network. A vertex labeling of a graph $G = (V, E)$ is an assignment $f$ of labels to the vertices of $G$ that induces for each edge $uv \in E(G)$ a label depending on the vertex labels $f(u)$ and $f(v)$. Most graph labeling trace their origin to one introduced by Rosa [5] in 1967 called as graceful labeling. Gnanajothi[2] in 1991 defined a graph $G$ with $q$ edges to be odd-graceful if there is an injection $f$ from $V(G)$ to $\{0, 1, 2, \ldots, 2q - 1\}$ such that, when each edge $xy$ is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, 2q - 1\}$. In 2012 Sridevi, Navaeethakrishnan, Nagarajan and Nagarajan[6] called a graph $G$ with $q$ edges odd-even graceful if there exist an injection $f$ from the vertices of $G$ to $\{1, 3, 5, \ldots, 2q + 1\}$ such that, when each edge $uv$ is assigned the label $|f(u) - f(v)|$, the resulting edge labels are $\{2, 4, 6, \ldots, 2q\}$.

Definition 1. A hairy cycle $C_n \odot mK_1$ is the graph obtained by joining $m$ pendent edges to each vertex of cycle $C_n$

2 Main Results

Theorem 2. Every cycle $C_n$ is odd-even graceful for $n \equiv 0, 3 \pmod{4}$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle $C_n$. Define $f : V(C_n) \rightarrow \{1, 3, \ldots, 2n + 1\}$ by

$f(v_{2i-1}) = 2(n + 1 - i) + 1 \ 1 \leq i \leq \lceil \frac{n}{2} \rceil$

$f(v_{2i}) = \begin{cases} 
2i - 1 & 1 \leq i \leq \lceil \frac{n}{4} \rceil \\
2i + 1 & \lceil \frac{n}{4} \rceil + 1 \leq i \leq \lceil \frac{n}{2} \rceil 
\end{cases}$

Case (i): $n \equiv 0 \pmod{4}$

Now we prove that $f$ is odd-even graceful labeling for $n \equiv 0 \pmod{4}$
Let
\[ S = \{ f(v_i) | 1 \leq i \leq n \} = \{ f(v_1), f(v_2), \ldots, f(v_{n/2 - 1}), f(v_{n/2}), f(v_{n/2 + 1}), \ldots, f(v_{n-1}), f(v_n) \} \]
\[ = \{ 2n + 1, 1, 2(n + 1 - 2) + 1, 3, 2(n + 1 - 3) + 1, 5, \ldots \} \]
\[ = \{ 2(n + 1 - n/4) + 1, 2(n/4) - 1, 2(n + 1 - (n/4 + 1) + 1, 2(n/4 + 1) + 1, \ldots (2n + 1 - (n - 2)/2) + 1, (n - 2 + 1), \]
\[ 2(n + 1 - n/2) + 1, 2(n/2 + 1) \} \]
\[ = \{ 2n + 1, 1, 2n - 1, 3, 2n - 3, 5, \ldots 3n/2 + 3, n/2 - 1, 3n/2 + 1, \]
\[ n/2 + 3, \ldots, n + 5, n - 1, n + 3, n + 1 \} \]
\[ = \{ 1, 3, \ldots, n/2 - 1, n/2 + 3, \ldots 3n/2 + 1, 3n/2 + 3, \ldots, n - 1, \]
\[ n + 1, n + 3, \ldots, 2n - 1, 2n + 1 \} \] (1)

Consider edge sets \( E_1, E_2, E_3 \) as follows
\[ E_1 = |f(v_n) - f(v_1)| = |(n + 1) - (2n + 1)| = n \]
\[ E_2 = \{ |f(v_{i+1}) - f(v_{i})|; 1 \leq i \leq \left[ \frac{n}{2} \right] \} = \{ |f(v_1) - f(v_2)|, |f(v_3) - f(v_4)|, \ldots, |f(v_{n-1}) - f(v_n)| \} \]
\[ = \{ |(2n+1)-1|, |(2n-1)+3|, \ldots |(n+5)-(n-1)|, |(n+3)-(n+1)| = \{ 2n, 2n - 4, 2n - 8, \ldots, 2, 4 \} \] \]
\[ E_3 = \{ |f(v_{i+1}) - f(v_{2i+1})|; 1 \leq i \leq \left[ \frac{n}{2} \right] - 1 \} = \{ |f(v_2) - f(v_3)|, |f(v_4) - f(v_5)|, \ldots, |f(v_{n-2}) - f(v_{n-1})| \} = \{ |1-(2n-1)|, |3-(2n-3)|, \ldots |(n-3)-(n+5)|, |(n+1)-(n+3)| = \{ 2n-2, 2n-6, \ldots, 8, 4 \} \}

Clearly
\[ E_1 \cup E_2 \cup E_3 = \{ 2, 4, 6, \ldots, (2n - 2), 2n \} \] (2)

From equation (1) and (2) it is clear that \( C_n \) for \( n \equiv 0 \mod 4 \) is odd-even graceful.

**Case (ii):** \( n \equiv 3 \mod 4 \) Proof same as in case (i). Therefore, \( C_n \)
for \( n \equiv 3 \mod 4 \) is odd-even graceful.

Hence, every cycle \( C_n \) is odd-even graceful for \( n \equiv 0, 3 \mod 4 \).

\[ \square \]

**Theorem 3.** Every hairy cycles \( C_n \odot mK_1 \) is odd-even graceful for \( n \equiv 0, 3 \mod 4 \).

**Proof.** Let \( u_1, u_2, u_3, \ldots, u_n \) be the vertices of the cycle \( C_n \) and \( u_1^1, u_2^2, \ldots, u_n^m \) be the vertices attached to each vertex of \( u_i \) of \( C_n \).
Therefore \(|V(C_n \odot mK_1)| = n(m + 1)\) and \(|E(C_n \odot mK_1)| = n(m + 1)\).

Define \(f : V(C_n \odot mK_1) \rightarrow \{1, 3, 5, \ldots, 2n(m + 1) + 1\}\) by

\[
f(u_{2i-1}) = \begin{cases} 
1 + 2(i - 1)(m + 1) & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
3 + 2(i - 1)(m + 1) + 1 & \left\lceil \frac{n}{2} \right\rceil \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor 
\end{cases}
\]

\[
f(u_{2i}) = 2(m + 1)(n - i + 1) - 2m + 1 & 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil 
\]

\[
f(u_{2j-1}^{(j)}) = 2n(m + 1) + 1 - 2(i - 1)(m + 1) - 2(j - 1) & 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, 1 \leq j \leq m 
\]

\[
f(u_{2j}^{(j)}) = \begin{cases} 
1 + 2(i - 1)(m + 1) + 2j & 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil, 1 \leq j \leq m \\
3 + 2(i - 1)(m + 1) + 2j & \left\lceil \frac{n}{2} \right\rceil \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, 1 \leq j \leq m 
\end{cases}
\]

**Case (i)** \(m \equiv 0 \mod 4\)

Let

\[
S = \{f(u_i)[1 \leq i \leq n; f(u_i^{(j)})][1 \leq i \leq n, 1 \leq j \leq m]\}
\]

\[
= \{f(u_1), f(u_2), \ldots, f(u_n), f(u_{11}), f(u_{12}), \ldots, f(u_{1m}), f(u_{21}), f(u_{22}), \ldots, f(u_{2m}), f(u_{n1}), f(u_{n2}), \ldots, f(u_{nm})\}
\]

\[
= \{1, 2n(m + 1) - 2m + 1, 1 + 2(m + 1), 2n(m + 1) - 4m - 1, \\
\ldots, (n - 2)(m + 1) + 3, (n + 2)(m + 1) - 2m + 1, \\
2n(m + 1) + 1, 2n(m + 1) - 1, \ldots, 2n(m + 1) - 2(m - 1), \\
3, 5, 7, \ldots, 1 + 2m, \ldots, (n - 2)(m + 1) + 5, (n - 2)(m + 1) + 7, \\
\ldots, (n - 2)(m + 1) + 2m + 3\}
\]

\[
= \{1, 3, 5, \ldots, 2n(m + 1) - 1, 2n(m + 1) + 1\}
\]

(3)

Let us consider the edge set \(E_1, E_2, E_3\) as follows

\[
E_1 = \{|f(u_i) - f(u_{i+1})|1 \leq i \leq n - 1\} = \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, \ldots, |f(u_{n-1}) - f(u_{n})|\}
\]

\[
= \{|1 - (2n(m + 1) - 2m + 1)|, |2n(m + 1) - 2m + 1 - (1 + 2(m + 1))|, \ldots, |(n - 2)(m + 1) + 3 - ((n + 2)(m + 1) - 2m + 1)|\}
\]
\[ E_2 = \{|f(u_i) - f(u_j)|, 1 \leq i \leq n, 1 \leq j \leq m\} = \{|f(u_1) - f(u_1^{(1)})|, |f(u_1) - f(u_1^{(2)})|, \ldots, |f(u_1) - f(u_1^{(m)})|, \ldots, |f(u_n) - f(u_n^{(1)})|, |f(u_n) - f(u_n^{(2)})|, \ldots, |f(u_n) - f(u_n^{(m)})|\} \]

\[ = \{|1 - (2n(m+1)+1)|, |1 - (2n(m+1) - 1)|, \ldots, |1 - (2n(m+1) - 2m+3)|, \ldots, |(n+2)(m+1) - 2m + 1 - ((n-2)(m+1) + 5)|, \ldots, |(n+2)(m+1) - 2m + 1 - ((n-2)(m+1) + 2m + 3)|\} \]

\[ = \{2n(m+1), 2n(m+1) - 2, \ldots, 2n(m+1) - 2m + 2, \ldots, 2m, 2m - 2, \ldots, 2\} \]

\[ E_3 = \{|f(u_1) - f(u_n)|\} = \{|1 - ((n+2)(m+1) - 2m + 1)|\} = (n+2)(m+1) - 2m \]

Clearly

\[ E_1 \cup E_2 \cup E_3 = \{2, 4, \ldots, 2m - 2, 2m, \ldots, 2n(m+1) - 2, 2n(m+1)\} \]

(4)

From equation (3) and (4) it is clear that hairy cycles \( C_n \odot mK_1 \) is odd-even graceful for \( n \equiv 0 \mod 4 \).

**Case (ii)** \( m \equiv 3 \mod 4 \) Proof is same as in case (i).

Hence, hairy cycles \( C_n \odot mK_1 \) are odd-even graceful for \( n \equiv 0, 3 \mod 4 \).

**Theorem 4.** One-point union of cycles \( C_m^{(n)} \) is odd-even graceful for \( m = 4 \).
Proof. Let $u_0^i, u_1^i, u_2^i, u_3^i$ be the vertices of the $i$-th cycle of $C_m^{(n)}$ and let $u_0^i = u$ for all $i$.

**Case (i):** $n$ is even. Define $f : V(C_m^{(n)}) \rightarrow \{1, 3, 5, \ldots, 8n + 1\}$ by $f(u_0) = 1$.

For $j = 1, 3, \ldots, n - 1$

$f(u_1^j) = (8n + 1) - 4(j - 1); f(u_2^j) = 3 + 4(j - 1); f(u_3^j) = (8n - 3) - 4(j - 1)$

For $j = 2, 4, 6, \ldots, n$

$f(u_1^j) = 5 + 4(j - 2); f(u_2^j) = (8n - 5) - 4(j - 2); f(u_3^j) = 9 + 4(j - 2)$

Let $S = \{f(u_0)\} \cup \{f(u_1^j) | 1 \leq j \leq n\} \cup \{f(u_2^j) | 1 \leq j \leq n\} \cup \{f(u_3^j) | 1 \leq j \leq n\}$

$$S = \{f(u_0)\} \cup \{f(u_1^j) | j = 1, 3, \ldots, n - 1\} \cup \{f(u_1^j) | j = 2, 4, \ldots, n\}$$

$$\cup \{f(u_2^j) | j = 1, 3, \ldots, n - 1\} \cup \{f(u_2^j) | j = 2, 4, \ldots, n\}$$

$$\cup \{f(u_3^j) | j = 1, 3, \ldots, n - 1\} \cup \{f(u_3^j) | j = 2, 4, \ldots, n\}$$

$$= \{f(u_0)\} \cup \{f(u_1^{(1)}), f(u_1^{(3)}), \ldots, f(u_1^{(n-1)})\} \cup \{f(u_1^{(2)}), f(u_1^{(4)}), \ldots, f(u_1^{(n)})\} \cup \{f(u_2^{(1)}), f(u_2^{(3)}), \ldots, f(u_2^{(n-1)})\} \cup \{f(u_2^{(2)}), f(u_2^{(4)}), \ldots, f(u_2^{(n)})\}$$

$$\cup \{f(u_3^{(1)}), f(u_3^{(3)}), \ldots, f(u_3^{(n-1)})\} \cup \{f(u_3^{(2)}), f(u_3^{(4)}), \ldots, f(u_3^{(n)})\}$$

$$= \{1\} \cup \{8n + 1, 8n - 7, 8n - 15, \ldots, 4n + 9\} \cup \{5, 13, 21, \ldots, 4n - 7, 4n - 3\} \cup \{3, 11, 19, \ldots, 4n - 9, 4n - 5\} \cup \{8n - 5, 8n - 13, \ldots, 4n + 11, 4n + 3\} \cup \{8n - 3, 8n - 11, \ldots, 4n + 5\}$$

$$\cup \{9, 17, \ldots, 4n - 7, 4n + 1\}$$

$$= \{1, 3, 5, 9, 11, 13, 17, \ldots, 4n - 5, 4n - 7, 4n - 9, \ldots, 8n - 5, 8n - 3, 8n + 1\}$$

(5)

Consider edge sets $E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8$ as follows

For $j = 1, 3, \ldots, n - 1$

$E_1 = \{|f(u_1^j) - f(u_0)|\} = \{|f(u_1^{(1)}) - f(u_0)|, |f(u_1^{(3)}) - f(u_0)|, \ldots, |f(u_1^{(n-1)}) - f(u_0)|\}$

$$= \{|(8n + 1) - 1|, |(8n - 1) - 1|, \ldots, |(4n + 9) - 1|\} = \{8n, 8n - 8, \ldots, 4n + 8\}$$

$E_2 = \{|f(u_1^j) - f(u_2^j)|\} = \{|f(u_1^{(1)}) - f(u_2^{(1)})|, |f(u_1^{(3)}) - f(u_2^{(3)})|, \ldots, |f(u_1^{(n-1)}) - f(u_2^{(n-1)})|\}$

$$= \{|(8n + 1) - 1|, |(8n - 1) - 1|, \ldots, |(4n + 9) - 1|\} = \{8n, 8n - 8, \ldots, 4n + 8\}$$
\[
E_3 = \{ |f(u_j^{(2)}) - f(u_j^{(3)})| \} = \{ |f(u_2^{(1)}) - f((u_3^{(1)})), |f(u_2^{(3)}) - f((u_3^{(3)}))| \} = \{ 3 - (8n - 3), 11 - (8n - 11), \ldots , 4n - 5 - (4n + 5) \} = \\
\{ 8n - 6, 8n - 22, \ldots , 10 \}
\]
\[
E_4 = \{ |f(u_3^{(j)}) - f(u_0)| \} = \{ |f(u_3^{(1)}) - f(u_0)|, |f(u_3^{(3)}) - f(u_0)|, \ldots , |f(u_3^{(n-1)}) - f(u_0)| \} = \\
\{ |(8n - 3) - 1|, |(8n - 11) - 1|, \ldots , |(4n + 5) - 1| \} = \{ 8n - 4, 8n - 12, \ldots , 4n + 4 \}
\]
For \( j = 2, 4, \ldots , n \)
\[
E_5 = \{ |f(u_4^{(j)}) - f(u_0)| \} = \{ |f(u_4^{(2)}) - f(u_0)|, |f(u_4^{(4)}) - f(u_0)|, \ldots , |f(u_4^{(n)}) - f(u_0)| \} = \\
\{ |(8n - 2| - 1|, |(8n - 16| - 1|, \ldots , |(4n + 6) - 1| \} = \{ 8n - 10, 8n - 26, \ldots , 6 \}
\]
\[
E_6 = \{ |f(u_6^{(j)}) - f(u_0)| \} = \{ |f(u_6^{(2)}) - f(u_0)|, |f(u_6^{(4)}) - f(u_0)|, \ldots , |f(u_6^{(n)}) - f(u_0)| \} = \\
\{ |(8n - 12) - 1|, |(8n - 24) - 1|, \ldots , |(4n + 8) - 1| \} = \{ 8n - 14, 8n - 30, \ldots , 2 \}
\]
\[
E_7 = \{ |f(u_7^{(j)}) - f(u_0)| \} = \{ |f(u_7^{(2)}) - f(u_0)|, |f(u_7^{(4)}) - f(u_0)|, \ldots , |f(u_7^{(n)}) - f(u_0)| \} = \\
\{ |(8n - 14) - 1|, |(8n - 28) - 1|, \ldots , |(4n + 10) - 1| \} = \{ 8, 16, \ldots , 4n \}
\]
Clearly
\[
E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_6 \cup E_7 \cup E_8 = \{ 2, 4, 6, \ldots , 4n, 4n + 2, \ldots , 8n - 2, 8n \}
\]
(6)

From equation (5) and (6) it is clear that one-point union of cycles
\( C_4^{(n)} \) are odd-even graceful for even \( n \).

Case (ii): \( n \) is odd. Define \( f : V(C_m^{(n)}) \rightarrow \{1, 3, 5, \ldots , 8n + 1 \} \) by
\[
f(u_0) = 1
\]
For \( j = 1, 3, \ldots , n - 1 \)
\[
f(u_j^{(1)}) = (8n + 1) - 4(j - 1); \ f(u_j^{(3)}) = 3 + 4(j - 1); \ f(u_j^{(5)}) = (8n - 3) - 4(j - 1)
\]
For \( j = 2, 4, 6, \ldots , n \)
\[
f(u_j) = 5 + 4(j - 2); \ f(u_i) = (8n - 5) - 4(j - 2); \ f(u_i) = 9 + 4(j - 2)
\]
Proof is same as in case (i) as the change is only in the range of values \( j \) takes. Hence, one-point union of cycles \( C_4^{(n)} \) are odd-even graceful.

![Figure 3: Odd-even graceful labeling of \( C_4^{(3)} \)](image)

References


