Performance Evaluation of a Single M/Geo/1 Queue Capable of Handling Two Like Jobs as a Single Entity

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Abstract

This paper introduces a queuing model named M/Geo/1 queue with transient behavior capable of handling two jobs arriving in succession within a time interval ∆t as a single entity. Both paired and unpaired jobs are executed. Arrival distribution is Poisson and general service discipline is a linear combination of geometric distribution. The paired and unpaired jobs are represented as two job types. The probability for the server to take more time to process paired jobs is derived. The mean and variance of the service time distribution of the server are derived. The cost equation is used to derive the relative measures of the model. The model is validated by comparing its efficiency with the standard M/G/2 model.

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Key Words and Phrases: Treating two like jobs as a single entity M/Geo/1 model, geometric distribution, M/G/2 model, probability mass function, time slicing, and average number of jobs in the system.

1 Introduction

In this paper a single class of jobs arriving to a service facility with a single server is considered. It is grouped into two categories, paired and unpaired jobs. Paired jobs are assumed to take longer processing time. The service distribution for both the jobs is geometric in nature. We analyze the model by comparing it with the G/M/2 model.

1.1 System Description

In a multitasking operating system, all jobs cannot run at once, so jobs must be batched / grouped up and then scheduled according to some policy. A queue might be a suitable option in this case. This paper proposes a new model M/Geo/1 queue where jobs arrive in accordance with Poisson process and the service is a mixture of geometric distribution. Two jobs that arrive in succession with negligible inter-arrival time are grouped to form a single paired job. The M/Geo/1 model assumes Poisson arrivals at rate $\lambda$. $\lambda E[z] < 1$ for existence of $W_q$, $L_q$, $L$ and $W$ where $W_q$ is average amount of time a customer spends in the queue, $L$ is average number of customers in the system and $L_q$ is the average number of customers waiting in the queue. A service distribution with probability $q^{2k}$ for paired jobs and $(1 - q^{2k})$ for unpaired jobs is considered. The jobs are served in the order of their arrival. If three jobs arrive at the same instant two are paired to form a paired job and the third is treated as an unpaired job. For an arbitrary job in the M/Geo/1 system.

$$\text{Jobs waiting in the queue} = \text{Work done in the system}$$

when the job arrives \hspace{1cm} (1)

Let the paired and unpaired jobs be represented by the random variables $X_1$ and $X_2$ respectively. A random variable $Z = g(X_1, X_2)$
is framed where $X_1$ and $X_2$ are both geometrically distributed with probability mass function $P(X_1 = x_1) = pq^{x_1-1}$ and $P(X_2 = x_2) = pq^{x_2-1}$. Here $x_1$ and $x_2$ denote the number of cycles (each of constant time) the job has to complete in order to get serviced.

Let $X_1$ take $k$ units of time to get processed then the probability that $X_2$ will require more time than $X_1$ is given by

$$P(X_1 = x_1, X_2 = x_2) = \sum_{x_1=k}^{\infty} pq^{x_1-1} \sum_{x_2=k+1}^{\infty} (pq^{x_2-1}) = q^{2k}$$  \hspace{1cm} (2)

The service distribution $Z$ is given by

$$Z = (1 - q^{2k})X_1 + q^{2k}X_2$$  \hspace{1cm} (3)

Let $x_1 \in X_1$ and $x_2 \in X_2$ denote the unpaired and paired arrivals to the server, then

$$Z = (1 - q^{2k})x_1 + q^{2k}x_2$$  \hspace{1cm} (4)

Both $X_1$ and $X_2$ are geometric variables with mean $1/p$.

Also $E[Z] = (1 - q^{2k})E[X_1] + q^{2k}E[X_2]$

$E[X_1] = E[X_2] = E[Z] = 1/p, p + q = 1$  \hspace{1cm} (5)

$$E[Z^2] = E\left[\left((1 - q^{2k})X_1 + q^{2k}X_2\right)^2\right]$$

$$E[Z^2] = (1 - q^{2k})^2 E[X_1^2] + q^{4k} E[X_2^2] + 2(1 - q^{2k})q^{2k} E[X_1X_2]$$  \hspace{1cm} (6)

$$E[X_1^2] = \sum_{x_1=2}^{\infty} x_1(x_1-1)pq^{x_1-1} + \frac{1}{p} = \frac{q+1}{p^2}$$

$$E[Z^2] = \frac{[(1 - q^{2k})^2 + q^{4k}]q + 1}{p^2}$$  \hspace{1cm} (7)

1.2 Literature Review

Mixture distributions and mixture models are of interest in the recent years. This has led to the development of M/G/k and G/M/k queuing models, which suit to model real time systems. In [1], it has been established that every geometric stable random variable can
be represented as a product of geometrically stable random variables. The authors in [2], derive the mathematical properties and procedures for estimation for a discrete exponentiated exponential distribution. A comparative study is made with recent discrete distribution using data sets on insurance. [3] introduces a new model named as the multivariate generalized Poisson log $-t$ geometric process model. Its application is established through an analysis of the possession and/or use of two drugs in New South Wales, Australia. P.Jodra and M.D. Jiminez-Gamero provide analytical solution for a logarithmic integral in terms of the Lerch transcendent function together with Sterling numbers of the first kind in [4]. Size based discrete phase type and matrix geometric distributions are considered in [5]. A study is made on their factorial moment distributions. A new distribution called Beta-Pareto-Geometric is introduced by M. Nassar and N. Nada in their work "A new generalization of the Pareto-geometric distribution." Its characteristics are derived and parameters are estimated using maximum likelihood and the information matrix is constructed. [7], generalizes geometric and binomial distributions of order $k$. The probability mass function is derived and a detail study of the distribution is made. [8], analyses two stochastic representations of multivariate geometric distributions using their memory-less property. Also a third stochastic construction based on non-decreasing random walk is explored. The Poisson geometric process model proposed in [9] is designed to overcome the problem of outliers. It describes the trend and the mixing parameters in the scale mixture representations detect the outlying observations in this model.
A model with patient and impatient type of customers is analyzed in [10]. The busy period of the server and the limiting distribution of the waiting time are derived in this paper.

1.3 Organization of the paper

In section 2 we discuss the proposed M/Geo/1 model for treating two like jobs as a single entity. The concept of work and cost equation are introduced using which the relative measures are derived and the average number of jobs executed during a busy period is found. In section 3 the standard M/G/k model is considered with $k = 2$. In section 4 a comparative study of the results obtained in
section 2 and 3 are carried out to validate the model. Section 5 gives the conclusions.

2 Single server model with transient behavior and geometric (mixture) service distribution

The operation of time sharing in computer systems with fixed time slice is a common feature. At the end of time slice the program would have completed execution with probability \( p \). This gives rise to a probability \( q = 1 - p > 0 \) that it needs to perform more computations. Treating two like jobs as a single entity M/Geo/1 model developed assumes Poisson arrivals and allows the service distribution as a linear combination of geometric distribution with probability mass function \( P(X_1 = x_1) = pq^{x_1-1} \) and \( P(X_2 = x_2) = pq^{x_2-1} \) respectively for paired and unpaired jobs. Here \( x_2 > x_1 \), \( x_1 \) denotes the number of time units (slices) for the completion of an unpaired job. That is the number of trials required for success corresponds to the number of executions of the same job each execution taking unit time (single slice). Similarly \( x_2 \) corresponds to the number of executions of a (same) paired job. To analyze this model we use the concept of work and cost equations related to it as given in [12].

**Definition:** The basic cost identity is given by

\[
\text{Average rate at which the system earns } = \lambda_a \times (\text{average amount an entering customer pays}) \tag{8}
\]

where \( \lambda_a \) is defined to be average arrival rate of entering customers.

**Definition: Cost rule:** each customer pays at a rate of \( y \)/unit time when his remaining service time is \( y \) whether he is in service or in queue. Thus the rate at which the system earns is just the work in the system. If \( V \) denotes the average work in the system then

\[
V = \lambda_a \times E[\text{Amount paid by a customer}] \tag{9}
\]
2.1 Performance measures

(a) $W_Q$: Waiting time in the queue

For an arbitrary job

$$E[\text{job's waiting in the queue}] = E[\text{work done in the system when the job arrives}]$$  (10)

The average work as seen by a job on its arrival will be equal to the time average work in the system. This is due to the Poisson nature of the job arrivals. Therefore

$$W_q = V = \lambda E[\text{Amount paid by a job}] = \lambda E[Z]W_q + \frac{\lambda}{2}E[Z^2]$$

$$W_q = \frac{\lambda \left\{ \left[ (1 - q^2)^2 + q^4k \right] q + 1 \right\}}{2p(p - \lambda)}.$$  (11)

(b) $L_q$: Average length of the queue = $\lambda W_q$.  (12)

(c) $W$: Average amount of time a job spends in the system

$$W = W_q + E[Z] = \frac{\lambda \left\{ \left[ (1 - q^2)^2 + q^4k \right] q + 1 \right\}}{2p(p - \lambda)} + \frac{1}{p}.$$  (13)

(d) $L$: Average number of customers in the system

$$L = \lambda W = \frac{\lambda^2 \left\{ \left[ (1 - q^2)^2 + q^4k \right] q + 1 \right\}}{2p(p - \lambda)} + \frac{\lambda}{p}.$$  (14)

3 M/G/2 model description

The model is derived from the standard M/G/k model by taking $k = 2$, that is the number of servers is two. The customers arrive at a Poisson rate $\lambda$ and are served by any one of the two servers. The service distribution is geometric in nature with probability mass function $P(S = s) = pq^{s-1}, s = 1, 2, 3, \ldots$. In this case, the work in the system when the job arrives = $2 \times \{\text{time the job spends in the queue} + \text{sum of the remaining service times of all other jobs in service at the moment when the arrival enters service}\}$. 
3.1 Relative measures

The relative measures are obtained by putting $k = 2$ in the standard M/G/k model discussed in [17]. Hence we get

$$E[S] = \frac{1}{p}$$  \hspace{1cm} (15)

$$E[S^2] = \frac{2q}{p^2}\cdot p + q = 1$$  \hspace{1cm} (16)

(a) $W'_Q$: Waiting time in the queue

$$W'_Q \approx \frac{\lambda^2 q}{(2p - \lambda)[(p + \lambda)(2p - \lambda) + 2\lambda^2]}.$$  \hspace{1cm} (17)

(b) $L'_Q$: Average length of the queue

$$L'_Q = \frac{\lambda^3 q}{(2p - \lambda)[(p + \lambda)(2p - \lambda) + 2\lambda^2]}.$$  \hspace{1cm} (18)

(c) $W'$: Average amount of time a job spends in the system

$$W' = W'_Q + E[S] = \frac{\lambda^2 q}{(2p - \lambda)[(p + \lambda)(2p - \lambda) + 2\lambda^2]} + \frac{1}{p}$$  \hspace{1cm} (19)

(d) $L'$: Average number of jobs in the system

$$L' = \lambda W' = \frac{\lambda^3 q}{(2p - \lambda)[(p + \lambda)(2p - \lambda) + 2\lambda^2]} + \frac{\lambda}{p}$$  \hspace{1cm} (20)

4 Numerical Results

4.1 Evaluation of relative measures when the arrival rate $\lambda = p - 0.009$ where $p$ takes values from 0.01 to 0.1 in increments of 0.01:

Let $\lambda$ be the arrival rate of the job which is Poisson in nature. The probability of the arrival of a unpaired job is $(1 - q^k)$ and that of paired jobs is $q^k$. The probability $p$ takes values from 0.01 to 0.1 in increments of .01. The arrival rate $\lambda$ is chosen in such a manner that $\lambda E[Z] < 1$. The mean value of the exponentially distributed
random variable is $1/p$, that is $\lambda < p$. This condition follows from the renewal theory which states that "if the server is always busy then the departure rate would be $1/E[Z]$ which must be larger than the arrival rate $\lambda$ to obtain finite values for the relative measures." For our numerical validation $\lambda$ is fixed throughout as $p-0.009$. To obtain the value of $k$ we need simulate $X$ such that $P[X = k] = p(1 - p)^{k-1}, k \geq 1$

Clearly $\sum_{k=1}^{j-1} P[X = k] = 1 - P\{X > (j - 1)\} = 1 - (1 - p)^{j-1}$

These geometrically distributed random variables are computed using the following algorithm.

1. Generate uniformly distributed random variable $U \in (0, 1)$ using MATLAB software.
2. As $1-U$ has the same distribution as $U$ define $X$ by
   
   $X = 1 + \left\lceil \frac{\log U}{\log(1-p)} \right\rceil = k$

3. Compute the value of the random variable $X$ from the value of $U$.

The relative measures are computed by using the formulas derived in section 2.1. For the $M/G/2$ model the average waiting time in the in the queue is obtained as an approximation given for the standard $M/G/k$ model from [12] and are shown in the table 1.

**Table 1**
Table when $U=0.3922$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$k$</th>
<th>Model</th>
<th>$W_q$</th>
<th>$L_q$</th>
<th>$W$</th>
<th>$L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0010</td>
<td>94.1277</td>
<td>M/Geo/1</td>
<td>0.0011</td>
<td>1.111e-006</td>
<td>100</td>
<td>0.100</td>
</tr>
<tr>
<td>0.0110</td>
<td>47.32619</td>
<td>M/Geo/1</td>
<td>0.0244</td>
<td>2.6883e-004</td>
<td>50.084</td>
<td>.5503</td>
</tr>
<tr>
<td>0.0210</td>
<td>31.7268</td>
<td>M/Geo/1</td>
<td>0.0700</td>
<td>0.0015</td>
<td>33.40</td>
<td>.7015</td>
</tr>
<tr>
<td>0.0310</td>
<td>23.9267</td>
<td>M/Geo/1</td>
<td>0.1377</td>
<td>0.0043</td>
<td>25.1377</td>
<td>.7793</td>
</tr>
<tr>
<td>0.0410</td>
<td>19.2463</td>
<td>M/Geo/1</td>
<td>0.2277</td>
<td>0.0093</td>
<td>20.2277</td>
<td>.8293</td>
</tr>
<tr>
<td>0.0510</td>
<td>16.1258</td>
<td>M/Geo/1</td>
<td>0.3398</td>
<td>0.0173</td>
<td>17.0065</td>
<td>.8673</td>
</tr>
<tr>
<td>0.0610</td>
<td>13.8966</td>
<td>M/Geo/1</td>
<td>0.4742</td>
<td>0.0289</td>
<td>14.7599</td>
<td>.9004</td>
</tr>
<tr>
<td>0.0710</td>
<td>12.2245</td>
<td>M/Geo/1</td>
<td>0.6307</td>
<td>0.0448</td>
<td>13.1307</td>
<td>.9323</td>
</tr>
<tr>
<td>0.0810</td>
<td>10.9237</td>
<td>M/Geo/1</td>
<td>0.8094</td>
<td>0.0656</td>
<td>11.9205</td>
<td>.9656</td>
</tr>
<tr>
<td>0.0910</td>
<td>9.8830</td>
<td>M/Geo/1</td>
<td>1.0103</td>
<td>0.0919</td>
<td>11.0103</td>
<td>1.0019</td>
</tr>
<tr>
<td></td>
<td></td>
<td>M/Geo/2</td>
<td>1.8291</td>
<td>.1665</td>
<td>11.8291</td>
<td>1.0765</td>
</tr>
</tbody>
</table>

5 Conclusion

From the table values it is found that the model developed performs at its best when the value of $p$ is very small. As $p$ increases $k$ decreases therefore the probability that a paired job will take more time for service decreases. This shows that the treating two like jobs as a single entity M/Geo/1 model gives better outputs than the G/M/2 model for very small values of $p$. In this case all the relative measures say $L, L_q, W, W_q$ are less for the model developed. Hence treating two like jobs as a single entity model gives the best output when the number of paired jobs is greater than the number of unpaired jobs. The limitations of the model is that number of paired job arrivals must be very high and the value of $p$ should be very small as $p$ increases the standard model gives better output.
than the developed model.

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References


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