BITOPOLOGICAL $g^{**}b$-CLOSED SETS

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Abstract: The aim of this paper is to introduce a new class of sets called ($i, j$)−$g^{**}b$-closed sets in bitopological spaces. Also we study some of its basic properties and investigate the relationship with the other existing closed sets in bitopological space.

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1. Introduction

A triple $(X, \tau_1, \tau_2)$ where $X$ is a non-empty set and $\tau_1$ and $\tau_2$ are topologies on $X$ is called a bitopological space and kelly [5] initiated the study of such spaces. In 1985, Fukutake [3] introduced the concept of $g$-closed sets in bitopological spaces and after that several authors turned their attention towards generalizations of various concepts of topology by considering bitopological spaces and Pauline Mary Helen and Jansi Rani [6] introduced $g^{**}b$-closed sets in topological spaces and investigated its relationship with the other types of closed sets. The purpose of the present paper is to define a new class of closed sets called ($i, j$)-$g^{**}b$-closed sets.

2. Preliminaries

**Definition 2.1.** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called:
(1) (i, j)-pre-open set [3] if \( A \subseteq \tau_i\text{-int}(\tau_j\text{-cl}(A)) \) and a (i, j)-pre-closed if \( \tau_j\text{-cl}(\tau_i\text{-int}(A)) \subseteq A \).

(2) (i, j)-semi-open set [3] if \( A \subseteq \tau_j\text{-cl}(\tau_i\text{-int}(A)) \) and (i, j)-semi-closed set if \( \tau_i\text{-int}(\tau_j\text{-cl}(\tau_i\text{-int}(A))) \subseteq A \).

(3) (i, j)-semi-pre-open set [3] if \( A \subseteq \tau_j\text{-cl}(\tau_i\text{-int}(\tau_j\text{-cl}(A))) \) and (i, j)-semi-pre-closed set if \( \tau_i\text{-int}(\tau_j\text{-cl}(\tau_i\text{-int}(A))) \subseteq A \).

(4) (i, j)-generalized closed set [3] if \( \tau_j\text{-cl}(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-open in \( X \).

(5) (i, j)-generalized semi-pre-closed set [4] if \( \tau_j\text{-spcl}(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-open in \( X \).

(6) (i, j)-g*-closed set [8] if \( \tau_j\text{-cl}(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-g-open in \( X \).

(7) (i, j)-g**-closed set [9] if \( \tau_j\text{-cl}(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-g*-open in \( X \).

(8) (i, j)-semi-generalized closed set [3] if \( \tau_j\text{-scl}(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-semi-open in \( X \).

(9) (i, j)-generalized semi-closed set [3] if \( \tau_j\text{-scl}(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-open in \( X \).

(10) (i, j)-generalized pre-closed set [3] if \( \tau_j\text{-pcl}(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-open in \( X \).

(11) (i, j)-\( \psi \)-closed set [4] if \( \tau_j\text{scl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-sg-open in \( X \).

(12) (i, j)-g* semi-closed set [7] if \( \tau_j\text{-scl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-gs-open in \( X \).

**Definition 2.2.** A subset \( A \) of a topological space \( (X, \tau) \) is called

(1) b-open set [1] if \( A \subseteq \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A)) \) and b-closed set if \( \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A)) \subseteq A \).

(2) g*b closed [10] if \( \text{bcl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \( X \).

(3) g*-closed set [10] if \( \text{cl}(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is g-open in \( X \).
3. (i, j)-g**b-Closed Sets

We introduce the following definitions.

**Definition 3.1.** A subset \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) is called \((i, j)\)-g**b closed if \( \tau_j-bcl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i - g^* \)-open in \( X \).

**Definition 3.2.** A subset \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) is called \((i, j)\)-g**b closed if \( \tau_j-bcl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i-g^* \)-open in \( X \).

**Theorem 3.3.** Every \( \tau_j \)-closed set is \((i, j)\)-g**b closed.

**Proof.** Proof follows from the definition. \( \square \)

**Remark 3.4.** Converse of the above theorem is not true in general as seen from the following example.

**Example 3.5.** Let \( X = \{a, b, c\} \), \( \tau_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\} \) and \( \tau_2 = \{\phi, X, \{a\}, \{a, b\}\} \). Here \( \{b\} \) and \( \{a, c\} \) are \((i, j)\)-g**b closed but not \( \tau_j \)-closed.

**Theorem 3.6.** Every \( \tau_j \)-b-closed set is \((i, j)\)-g**b closed.

**Proof.** Let \( A \) be \( \tau_j \)-b-closed in \( X \) such that \( A \subseteq U \) and \( U \) be \( \tau_i-g^* \)-open. Since \( A \) is \( \tau_j \)-b-closed, \( \tau_j-b \text{cl}(A) \subseteq U \). Therefore \( A \) is \((i, j)\)-g**b closed. \( \square \)

**Remark 3.7.** Converse of the above theorem is not true in general as seen from the following example.

**Example 3.8.** Let \( X = \{a, b, c\} \), \( \tau_1 = \{\phi, X, \{a\}, \{b\}\} \) and \( \tau_2 = \{\phi, X, \{b, c\}\} \). Clearly, the set \( \{a, b\} \) is \((i, j)\)-g**b-closed but not \( \tau_j \)-b closed.

**Theorem 3.9.** Every \((i, j)\)-g*-closed set is \((i, j)\)-g**b-closed but not conversely.

**Proof.** Let \( A \) be \((i, j)\)-g*-closed in \( X \) such that \( A \subseteq U \) and \( U \) be \( \tau_i-g^* \)-open. Since every \( \tau_i-g^* \)-open is \( \tau_i-g \)-open, \( U \) is \( \tau_i-g \)-open. Therefore \( \tau_j-cl(A) \subseteq U \). But \( \tau_j-b cl(A) \subseteq \tau_j-cl(A) \subseteq U \). Therefore \( \tau_j-b cl(A) \subseteq U \). Hence \( A \) is \((i, j)\)-g**b-closed. \( \square \)

**Example 3.10.** Let \( X = \{a, b, c\} \), \( \tau_1 = \{\phi, X, \{a, b\}\} \) and \( \tau_2 = \{\phi, X, \{c\}\} \). Here the set \( \{a\} \) and \( \{b\} \) are \((i, j)\)-g**b closed but not \((i, j)\)-g*-closed.

**Theorem 3.11.** Every \((i, j)\)-g** closed set is \((i, j)\)-g**b closed.
Proof. Let A be \((i, j)-g^{**}\)-closed in \(X\) such that \(A \subseteq U\) and \(U\) be \(\tau_i-g^*\)-open. Since A is \((i, j)-g^{**}\)-closed, \(\tau_j-\text{cl}(A) \subseteq U\). But \(\tau_j-\text{b cl}(A) \subseteq \tau_j-\text{cl}(A) \subseteq U\). Hence A is \((i, j)-g^{**}\)-b-closed.

Remark 3.12. Converse of the above theorem is not true in general as seen from the following example.

Example 3.13. Let \(X = \{a, b, c\}\), \(\tau_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}\) and \(\tau_2 = \{\phi, X, \{a\}, \{a, b\}\}\). Here the set \(\{b\}\) is \((i, j)-g^{**}\)-b closed but not \((i, j)-g^{**}\)-closed.

Theorem 3.14. Every \((i, j)-g^*\)-b closed set is \((i, j)-g^{**}\)-b closed.

Proof. Let A be \((i, j)-g^{**}\)-b closed in \(X\) such that \(A \subseteq U\) and \(U\) be \(\tau_i-g^*\)-open. Since every \(\tau_i-g^*\)-open set is \(\tau_i-g^*\)-open, \(\tau_j-\text{p cl}(A) \subseteq U\). Therefore A is \((i, j)-g^{**}\)-b-closed.

Remark 3.15. The converse of the above theorem is not true in general as seen from the following example.

Example 3.16. Let \(X = \{a, b, c\}\), \(\tau_1 = \{\phi, X, \{a\}\}\) and \(\tau_2 = \{\phi, X, \{a\}, \{a, b\}\}\). Here the set \(\{c\}\) and \(\{a, c\}\) are \((i, j)-g^{**}\)-b closed but not \((i, j)-g^{**}\)-b closed.

Theorem 3.17. Every \((i, j)-g^{**}\)-b closed set is \((i, j)-g^p\)-closed but not conversely.

Proof. Let A be \((i, j)-g^{**}\)-b closed in \(X\) such that \(A \subseteq U\) and \(U\) be \(\tau_i\)-open. Since every \(\tau_i\)-open set is \(\tau_i-g^*\)-open, \(\tau_j-\text{cl}(A) \subseteq U\). But \(\tau_j-\text{p cl}(A) \subseteq \tau_j-\text{cl}(A) \subseteq U\). Therefore \(\tau_j-\text{p cl}(A) \subseteq U\). Hence A is \((i, j)-g^p\)-closed.

Example 3.18. Let \(X = \{a, b, c\}\), \(\tau_1 = \{\phi, X, \{a\}\}\) and \(\tau_2 = \{\phi, X, \{a, b\}\}\). Clearly the set \(\{b\}\), \(\{b, c\}\) are \((i, j)-g^p\) closed but not \((i, j)-g^{**}\)-b closed.

Theorem 3.19. Every \((i, j)-g^{**}\)-b closed set is \((i, j)-g^s\)-closed.

Proof. Let A be \((i, j)-g^{**}\)-b closed in \(X\) such that \(A \subseteq U\) and \(U\) be \(\tau_i\)-open. Since every \(\tau_i\)-open set is \(\tau_i-g^*\)-open, \(\tau_j-\text{cl}(A) \subseteq U\). But \(\tau_j-\text{scl}(A) \subseteq \tau_j-\text{bcl}(A) \subseteq U\). Therefore \(\tau_j-\text{scl}(A) \subseteq U\). Hence A is \((i, j)-g^s\)-closed.

Remark 3.20. The converse of the above theorem is not true in general as seen form the following example.
Example 3.21. Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, X, \{a, b, d\}, \{d\}\}$ and $\tau_2 = \{\phi, X, \{b, c\}, \{d\}, \{b, c, d\}\}$. Clearly the set $\{d\}$ is $(i,j)$-gs-closed but not $(i,j)$-g**b-closed.

Theorem 3.22. Every $(i,j)$-g**b-closed set is $(i,j)$-gsp-closed but not conversely.

Proof. Let $A$ be $(i,j)$-g**b-closed in $X$ such that $A \subseteq U$ and $U$ be $\tau_i$-open. Since every $\tau_i$-open set is $\tau_i$-g*-open, $\tau_j$-$b \cl(A) \subseteq U$. But $\tau_j$-$sp \cl(A) \subseteq \tau_j$-$b \cl(A) \subseteq U$. Therefore $\tau_j$-$sp \cl(A) \subseteq U$. Hence $A$ is $(i,j)$-gsp-closed.

Example 3.23. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{b\}, \{a, c\}\}$ and $\tau_2 = \{\phi, X, \{b, c\}\}$. Clearly the set $\{b\}$, $\{c\}$ and $\{a,c\}$ are $(i,j)$-gsp-closed but not $(i,j)$-g**b-closed. Hence every $(i,j)$-gsp-closed set need not be $(i,j)$-g**b-closed.

Remark 3.24. $(i,j)$-pre-closedness and $(i,j)$-g**b-closedness are independent as seen from the following example.

Example 3.25. Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, X, \{b, c\}, \{d\}\}$ and $\tau_2 = \{\phi, X, \{b, c\}\}$. Here $\{b\}$, $\{c\}$ and $\{b, c\}$ are $(i,j)$-pre-closed but not $(i,j)$-g**b closed and $\{a, d\}$, $\{a, b, d\}$ and $\{a, c, d\}$ are $(i,j)$-g**b closed but not $(i,j)$-pre-closed.

Remark 3.26. $(i,j)$-semi-preclosedness and $(i,j)$-g**b-closedness are independent and is explained in the following examples.

Example 3.27. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a, c\}\}$, $\tau_2 = \{\phi, X, \{c\}, \{b, c\}\}$, $\sigma_1 = \{\phi, X\}$ and $\sigma_2 = \{\phi, X, \{a, b\}\}$. Clearly the set $\{c\}$ is $(i,j)$-semi-pre-closed but not $(i,j)$-g**b-closed.

Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{b, a\}\}$. Here $\{a, c\}$ is $(i,j)$-g**b closed but not $(i,j)$-semi-preclosed.

Remark 3.28. $(i,j)$-semi-closedness and $(i,j)$-g**b-closedness are independent and is shown below.

Example 3.29. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{\phi, X, \{a\}, \{b, c\}\}$. Then $(i,j)$-g**b closed $= \{\phi, X, \{a\}, \{c\}, \{b, c\}, \{a, c\}\}$ and $(i,j)$-semi-closed $= \{\phi, X, \{a\}, \{b\}, \{c\}, \{b, c\}\}$. Here $\{b\}$ is $(i,j)$-semi-closed but not $(i,j)$-g**b closed and $\{a, c\}$ is $(i,j)$-g**b-closed but not $(i,j)$-semi-closed.

Remark 3.30. $(i,j)$-sg-closedness and $(i,j)$-g**b-closedness are independent as shown in the following examples.
Example 3.31. Let \( X = \{a, b, c, d\} \), \( \tau_1 = \{\phi, X, \{a, b, d\}\} \) and \( \tau_2 = \{\phi, X, \{c\}, \{b, d\}, \{b, c, d\}\} \). The sets \( \{b\}, \{d\}, \{a, b\}, \{a, d\} \) and \( \{b, d\} \) are (\( i, j \))-sg-closed but not (\( i, j \))-\( g^* \)-b-closed.

Let \( X = \{a, b, c, d\} \), \( \tau_1 = \{\phi, X, \{a, b, d\}\} \) and \( \tau_2 = \{\phi, X, \{b, c\}, \{d\}, \{b, c, d\}\} \). The \( \{c\} \) and \( \{a, c\} \) are (\( i, j \))-\( g^* \)-b-closed but not (\( i, j \))-sg-closed.

Remark 3.32. (\( i, j \))-\( \psi \)-closedness and (\( i, j \))-\( g^* \)-b-closedness are independent. It can be seen from the following example.

Example 3.33. Let \( X = \{a, b, c, d\} \), \( \tau_1 = \{\phi, X, \{a\}, \{a, b\}, \{d\}\} \) and \( \tau_2 = \{\phi, X, \{c\}, \{a, d\}\} \). Clearly the set \( \{c\} \) is (\( i, j \))-\( \psi \)-closed but not (\( i, j \))-\( g^* \)-b-closed and the set \( \{c, d\} \) is (\( i, j \))-\( g^* \)-b-closed but not (\( i, j \))-\( \psi \)-closed.

Remark 3.34. (\( i, j \))-\( g^* \)-s-closedness and (\( i, j \))-\( g^* \)-b-closedness are independent as it can be seen from the following example.

Example 3.35. Let \( X = \{a, b, c\} \), \( \tau_1 = \{\phi, X, \{a\}, \{b, c\}\} \) and \( \tau_2 = \{\phi, X, \{b\}, \{b, c\}\} \). Clearly the set \( \{a, b\}, \{a, c\} \) and \( \{b, c\} \) are (\( i, j \))-\( g^* \)-b-closed but not (\( i, j \))-\( g^* \)-s-closed and the set \( \{d\} \) is (\( i, j \))-\( g^* \)-s-closed but not (\( i, j \))-\( g^* \)-b-closed.

Theorem 3.36. A set \( A \) is (\( i, j \))-\( g^* \)-b-closed if \( \tau_j\)-bcl(\( A \)) = \( A \) contains no nonempty \( \tau_i\)-\( g^* \)-closed set.

Proof. Let \( F \) be a \( \tau_i\)-\( g^* \)-closed set of \( (X, \tau_1, \tau_2) \) such that \( F \subseteq \tau_j\)-bcl(\( A \)) = \( A \). Then \( A \subseteq X - F \). Since \( A \) is (\( i, j \))-\( g^* \)-b-closed and \( X - F \) is \( \tau_i\)-\( g^* \)-open, \( \tau_j\)-bcl(\( A \)) \( \subseteq \) \( A \subseteq X - F \). This implies \( F \subseteq X - \tau_j\)-bcl(\( A \)). So \( F \subseteq (X - \tau_j\)-bcl(\( A \)) \cap (\( \tau_j\)-bcl(\( A \)) = \( \phi \)). Therefore \( F = \phi \). Therefore the set \( A \) is (\( i, j \))-\( g^* \)-b-closed if \( \tau_j\)-bcl(\( A \)) = \( A \) contains no nonempty \( \tau_i\)-\( g^* \)-closed set.

Theorem 3.37. If \( A \) is both \( \tau_i\)-\( g^* \)-open and (\( i, j \))-\( g^* \)-b-closed set of \( X \), then \( A \) is \( \tau_j\)-b-closed.

Proof. Since \( A \) is both \( \tau_i\)-\( g^* \)-open and (\( i, j \))-\( g^* \)-b-closed set in \( X \), \( \tau_j\)-bcl(\( A \)) \( \subseteq \) \( A \). Therefore \( \tau_j\)-bcl(\( A \)) = \( A \). Hence \( A \) is \( \tau_j\)-b-closed.

Theorem 3.38. If \( A \) is (\( i, j \))-\( g^* \)-b-closed and \( A \subseteq B \subseteq \tau_j\)-bcl(\( A \)), then \( B \) is (\( i, j \))-\( g^* \)-b-closed.

Proof. Let \( U \) be a \( \tau_i\)-\( g^* \)-open set of \( X \) such that \( B \subseteq U \). Then \( A \subseteq U \). Since \( A \) is (\( i, j \))-\( g^* \)-b-closed, then \( \tau_j\)-bcl(\( A \)) \( \subseteq \) \( U \). Now \( \tau_j\)-bcl(\( B \)) \( \subseteq \) \( \tau_j\)-bcl(\( \tau_j\)-bcl(\( A \))) = \( \tau_j\)-bcl(\( A \)) \( \subseteq \) \( U \). Therefore \( B \) is (\( i, j \))-\( g^* \)-b-closed in \( X \).
**Theorem 3.39.** Let $A \subseteq Y \subseteq X$ and suppose that $A$ is $(i,j)$-g**b**-closed in $X$, then $A$ is $(i,j)$-g**b**-closed relative to $Y$.

**Proof.** Given that $A \subseteq Y \subseteq X$ and $A$ is $(i,j)$-g**b**-closed in $X$. To show that $A$ is $(i,j)$-g**b**-closed relative to $Y$. Let $A \subseteq Y \cap U$ where $U$ is $\tau_i$-open in $X$. Since $A$ is $(i,j)$-g**b**-closed, $A \subseteq U$ implies $\tau_j$-b cl$(A)$. It follows that $Y \cap \tau_j$-b cl$(A) \subseteq Y \cap U$. Thus $A$ is $(i,j)$-g**b**-closed relative to $Y$.

**Theorem 3.40.** If $B \subseteq A \subseteq X$, $B$ is $(i,j)$-g**b**-closed relative to $A$ and $A$ is both $\tau_i$-open and $(i,j)$-g**b**-closed subset of $X$, then $B$ is $(i,j)$-g**b**-closed relative to $X$.

**Proof.** Let $B \subseteq G$ and $G$ be an $\tau_i$-open set in $X$. But given that $B \subseteq A \subseteq X$, therefore $B \subseteq A$ and $B \subseteq G$. This implies $B \subseteq A \cap G$. Since $B$ is $B$ is $(i,j)$-g**b**-closed relative to $A$, $A \cap \tau_j$-b cl$(B) \subseteq A \cap G$. Hence $A \cap \tau_j$-b cl$(B) \subseteq G$. Thus $(A \cap \tau_j$-b cl$(B)) \cup (\tau - j$-b cl$(B))^c \subseteq G \cup (\tau_j$-b cl$(B))^c$. Since $A$ is $(i,j)$-g**b**-closed in $X$, we have $\tau_j$-b cl$(A) \subseteq G \cup (\tau_j$-b cl$(B))^c$. Also $B \subseteq A$ implies $\tau_j$-b cl$(B) \subseteq \tau_j$-b cl$(A)$. Thus $\tau_j$-b cl$(B) \subseteq \tau_j$-b cl$(A) \subseteq G \cup (\tau_j$-b cl$(B))^c$. Therefore $\tau_j$-b cl$(B) \subseteq G$. Since $\tau_j$-b cl$(B)$ is not contained in $(\tau_j$-b cl$(B))^c$. Thus $B$ is $(i,j)$-g**b**-closed relative to $X$.

From the above discussions, we have the following diagram.

![Diagram](image-url)

where

1. $(i,j$-g**b**-closed
2. $\tau_j$-closed
3. $\tau_i$-open
4. $\tau_j$-open
5. $\tau_i$-closed
6. $\tau_j$-closed
7. $\tau_j$-open
8. $\tau_i$-closed
9. $(i,j$-s**g**-closed
10. $(i,j$-g**p**-closed
3. \( \tau_j \)-b-closed

4. \( (i, j) \)-gp-closed

5. \( (i, j) \)-ψ-closed

6. \( (i, j) \)-pre-closed

7. \( (i, j) \)-semi-closed

8. \( (i, j) \)-semi-pre-closed

11. \( (i, j) \)-gs-closed

12. \( (i, j) \)-g\(^{*}\)s-closed

13. \( (i, j) \)-g\(^{*} \)c-closed

14. \( (i, j) \)-g\(^{**}\)c-closed

15. \( (i, j) \)-g\(^{*}b\)-closed

References


