ON INTUITIONISTIC L-FUZZY IDEALS OF BP-ALGEBRAS

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Abstract: In papers [2, 3, 4, 5, 6] we studied the notion of fuzzy structures, fuzzy subalgebras, fuzzy ideals and L-fuzzy subalgebras and T-ideals of BP-Algebras.

Motivated by the works of several researchers on Intuitionistic fuzzy structures on Algebras such as BCI, BCK, BG and BF algebras, in this paper we introduce the notion of Intuitionistic L-fuzzy ideals in BP-algebras.

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1. Introduction

In 1965, Zadeh\cite{16} introduced the notion of fuzzy sets. In 1986, Atanassov\cite{1} generalized the concept of fuzzy sets into intuitionistic fuzzy sets. Goguen\cite{7} generalized the concept of fuzzy a sets into L-fuzzy sets where Lis any partially ordered algebraic structure. Motived by the works of Rosenfeld\cite{13} many mathematicians fuzzified several algebras.

In 1966, Inmai and Iseki\cite{8} introduced the notion of BCK and BCI algebras. Neggers and Kim\cite{12} developed the concept of $\beta$- algebras. In 2012, Sun Shin Ahn and Han\cite{14} introduced the concept of BP-Algebras.

2. Preliminaries

In this section we recall some basic definitions that are needed for our work.

**Definition 2.1.** A BP-algebra $(X, *, 0)$ is a non empty set $X$ with a constant $0$ and a binary operation $*$ satisfying the following conditions:

1. $x * x = 0$
2. $x * (x * y) = y$
3. $(x * z) * (y * z) = x * y$ for any $x, y \in X$

**Definition 2.2.** A fuzzy subset $\mu$ of a BP-algebra $(X, *, 0)$ is called a fuzzy BP-subalgebra of $X$ if, for all $x, y \in X$ the following condition is satisfied $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$

**Example 2.3.** Let $X = \{0, 1, 2, 3\}$ be a set with the following table

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>1</td>
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<td>1</td>
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<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Define $\mu : X \to [0, 1]$ by

\[
\mu(x) = \begin{cases} 
0.8 & \text{if } x = 0 \\
0.5 & \text{if } x = 2 \\
0.4 & \text{if } x = 1, 3 
\end{cases}
\]

Then $\mu$ is a Fuzzy BP-subalgebra of $X$. 
Definition 2.4. An Intuitionistic Fuzzy Subset (IFS) \( A \) in a non-empty set \( X \) is defined as an object of the form \( A = \{ < x, \mu_A(x), \nu_A(x) > / x \in X \} \) where \( \mu_A : X \rightarrow [0,1] \) is the degree membership and \( \nu_A : X \rightarrow [0,1] \) is the degree non-membership of the element \( x \in X \) satisfying \( 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \).

Definition 2.5. Let \( (L, \leq) \) be a complete lattice with least element 0 and greatest element 1 and an involutive order reversing operation \( N : L \rightarrow L \). Then an Intuitionistic L-fuzzy subset (ILFS) \( A \) in a non-empty set \( X \) is defined as an object of the form \( A = \{ < x, \mu_A(x), \nu_A(x) > / x \in X \} \) where \( \mu_A : X \rightarrow L \) is the degree membership and \( \nu_A : X \rightarrow L \) is the degree of non-membership of the element \( x \in X \) satisfying \( \mu_A(x) \leq N(\nu_A(x)) \).

3. INTUITIONISTIC L-FUZZY SUBALGEBRA OF BP-ALGEBRAS

In this section, we introduce the notion of Intuitionistic L-fuzzy BP-subalgebra.

Definition 3.1. An Intuitionistic Fuzzy Subset \( A \) in a BP-Algebra \( X \) is said to be an Intuitionistic Fuzzy Sub-Algebra of \( X \) if

1. \( \mu(x \ast y) \geq \min \{\mu(x), \mu(y)\} \)
2. \( \nu(x \ast y) \leq \max \{\nu(x), \nu(y)\} \forall x, y \in X \).

Definition 3.2. A L-Fuzzy subset \( \mu \) in a BP-Algebra \( X \) is said to be an L-fuzzy subalgebra of \( X \) if \( \mu(x \ast y) \geq \mu(x) \land \mu(y) \forall x, y \in X \).

Definition 3.3. An Intuitionistic L-Fuzzy Subset \( A \) in a BP-Algebra \( X \) is said to be an Intuitionistic L-Fuzzy Sub-Algebra of \( X \) if

1. \( \mu_A(x \ast y) \geq \mu_A(x) \land \mu_A(y) \)
2. \( \nu_A(x \ast y) \leq \nu_A(x) \lor \nu_A(y) \forall x, y \in X \).

Example 3.4. Consider the BP-Algebra \( X = \{0,1,2,3\} \) as in example 2.3. The Intuitionistic L-Fuzzy Subset \( A = \{ < x, \mu_A(x), \nu_A(x) > / x \in X \} \) of \( X \) given by

\[
\mu_A(x) = \begin{cases} 
0.6 & \text{if } x \neq 2 \\
0.1 & \text{if } x = 2 
\end{cases}
\]

and

\[
\nu_A(x) = \begin{cases} 
0.2 & \text{if } x \neq 2 \\
0.8 & \text{if } x = 2 
\end{cases}
\]

is an Intuitionistic L-Fuzzy BP-Subalgebra of \( X \).
Definition 3.5. An intuitionistic L-fuzzy subset $A$ of a BP-algebra $X$ is said to be an intuitionistic fuzzy BP-ideal of $X$ if,

1. $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$
2. $\mu_A(x) \geq \min\{\mu_A(x \ast y), \mu_A(y)\}$
3. $\nu_A(x) \leq \max\{\nu_A(x \ast y), \nu_A(y)\}$ for all $x, y \in X$.

Example 3.6. Let $X = \{0, a, b\}$ be a BP-algebra with Cayley table given by

\[
\begin{array}{ccc}
* & 0 & a & b \\
0 & 0 & b & a \\
a & a & 0 & b \\
b & b & a & 0
\end{array}
\]

Define $\mu_A(0) = 1$ and $\mu_A(a) = \mu_A(b) = t$, and $\nu_A(0) = 0$ and $\nu_A(a) = \nu_A(b) = s$, where $t, s \in (0, 1)$ and $s + t \leq 1$. Then $(\mu_A, \nu_A)$ is an Intuitionistic BP-ideal of $X$.

Definition 3.7. An intuitionistic L-fuzzy subset $A$ of a BP-algebra $X$ is said to be an intuitionistic L-fuzzy closed BP-ideal of $X$ if,

1. $\mu_A(0 \ast x) \geq \mu_A(x)$
2. $\nu_A(0 \ast x) \leq \nu_A(x)$
3. $\mu_A(x) \geq \mu_A(x \ast y) \land \mu_A(y)$
4. $\nu_A(x) \leq \nu_A(x \ast y) \lor \nu_A(y)$ for all $x, y \in X$.

Definition 3.8.
1. For any two fuzzy sets $\lambda$ and $\mu$ of a set $X$, their Cartesian product is defined to be the set $\lambda \times \mu : X \times X \to [0, 1]$ where $(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(y)\} \forall x, y \in X$.

2. For any two L-fuzzy sets $A$ and $B$ of $X$, their Cartesian product is defined to be the set $A \times B = (X \times X, \mu_A \times \mu_B)$ with $\mu_A \times \mu_B : X \times X \to L$ such that $(\mu_A \times \mu_B)(x, y) = \mu_A(x) \land \mu_B(y) \forall x, y \in X$.

Lemma 3.9. In an Intuitionistic L-fuzzy BP-subalgebra $A$ of $X$ we have

1. $\mu_A(0) \geq \mu_A(x)$
2. $\nu_A(0) \leq \nu_A(x) \forall x \in X$.

Proof.

$\mu_A(0) = \mu_A(x \ast x) \geq \mu_A(x) \land \mu_A(x) = \mu_A(x)$.

Similarly,

$\nu_A(0) = \nu_A(x \ast x) \leq \nu_A(x) \lor \nu_A(x) = \nu_A(x)$.

Theorem 3.10. Intersection of any two Intuitionistic L-fuzzy BP subalgebras of X is again an Intuitionistic L-fuzzy subalgebra of X.

Proof Let $A = \{< x, \mu_A(x), \nu_A(x) > / x \in X\}$ and $B = \{< x, \mu_B(x), \nu_B(x) > / x \in X\}$ be Intuitionistic L-fuzzy BP subalgebras of X and let $C = A \cap B$ Now, for every $x, y \in X$

$$\begin{align*}
\mu_C(x \ast y) &= \mu_{A \cap B}(x \ast y) \\
&= \mu_A(x \ast y) \land \mu_B(x \ast y) \\
&\geq (\mu_A(x) \land \mu_A(y)) \land (\mu_B(x) \land \mu_B(y)) \\
&= (\mu_A(x) \land \mu_B(x)) \land (\mu_A(y) \land \mu_B(y)) \\
&= \mu_C(x) \land \mu_C(y) \\
\nu_C(x \ast y) &= \nu_{A \cap B}(x \ast y) \\
&= \nu_A(x \ast y) \lor \nu_B(x \ast y) \\
&\leq (\nu_A(x) \lor \nu_A(y)) \lor (\nu_B(x) \lor \nu_B(y)) \\
&= (\nu_A(x) \lor \nu_B(x)) \lor (\nu_A(y) \lor \nu_B(y)) \\
&= \nu_C(x) \lor \nu_C(y)
\end{align*}$$

Hence, $C$ is an Intuitionistic L-fuzzy BP subalgebra of X.

We can generalize the above result as

Theorem 3.11. Intersection of any family Intuitionistic L-fuzzy BP subalgebras of X is again an Intuitionistic L-fuzzy subalgebra of X.

In the same way and by the definition of A we can prove the following

Theorem 3.12. If $A$ is an Intuitionistic L-Fuzzy BP subalgebra of X, then $\overline{A}$ is an Intuitionistic L-Fuzzy BP subalgebra of X.

Theorem 3.13. An Intuitionistic L-Fuzzy Subset $A$ of X is an Intuitionistic L-Fuzzy BP SubAlgebra of X if and only if the L-fuzzy subsets $\mu_A$ and $\overline{\nu}_A$ are L-fuzzy BP subalgebras of X.

Proof Let $A = \{x, \mu_A(x), \nu_A(x) : x \in X\}$ be an Intuitionistic L-fuzzy BP subalgebra of X. Clearly, $\mu_A$ is a L-fuzzy BP subalgebra of X. For all $x, y \in X$,

$$\nu_A(x \ast y) = 1 - \nu_A(x \ast y)$$
\[\begin{align*}
\geq & \quad 1 - [\nu_A(x) \lor \nu_A(y)] \\
= & \quad (1 - \nu_A(x)) \land (1 - \nu_A(y)) \\
= & \quad \tau(x) \land \tau(y)
\end{align*}\]

This proves that \(\tau\) is a \(L\)-fuzzy BP subalgebra of \(X\).

Conversely, assume \(\mu_A\) and \(\nu_A\) are \(L\)-fuzzy subalgebras of \(X\).

Then we have, \(\mu_A(x \ast y) \geq \mu_A(x) \land \mu_A(y)\) and \(\nu_A(x \ast y) \geq \nu_A(x) \land \nu_A(y)\) \(\forall x, y \in X\).

Hence, to prove that \(A = \{x, \mu_A(x), \nu_A(x) : /x \in X\}\) is an Intuitionistic \(L\)-fuzzy BP subalgebra of \(X\), it is enough to prove that \(\nu_A(x \ast y) \leq \nu_A(x) \land \nu_A(y)\) \(\forall x, y \in X\).

Since \(\nu_A\) is \(L\)-fuzzy subalgebra of \(X\),

\[
\begin{align*}
\nu_A(x \ast y) & \geq \nu_A(x) \lor \nu_A(y) \\
1 - \nu_A(x \ast y) & \geq (1 - \nu_A(x)) \land (1 - \nu_A(y)) \\
& = 1 - [\nu_A(x) \lor \nu_A(y)]
\end{align*}
\]

That is, \(\nu_A(x \ast y) \leq \nu_A(x) \lor \nu_A(y)\) \(\forall x, y \in X\).

This completes the Proof.

4. INTUITIONSTIC \(L\)-FUZZY BP-IDEALS

In this section, we introduce the notion of Intuitionistic \(L\) fuzzy BP-ideals and prove some simple theorems.

**Definition 4.1.** An intuitionistic \(L\)-fuzzy subset \(A\) of a BP-algebra \(X\) is said to be an intuitionistic \(L\)-fuzzy BP-ideal of \(X\) if,

1. \(\mu_A(0) \geq \mu_A(x)\) and \(\nu_A(0) \leq \nu_A(x)\)
2. \(\mu_A(x) \geq \mu_A(x \ast y) \land \mu_A(y)\)
3. \(\nu_A(x) \leq \nu_A(x \ast y) \lor \nu_A(y)\) for all \(x, y \in X\).

**Proposition 4.2.** For every intuitionstic \(L\)-fuzzy BP-subalgebra \(A = (\mu_A, \nu_A)\) in \(X\) we have the following properties:

1. \(\mu_A(0) \geq \mu_A(x)\)
2. \(\nu_A(0) \leq \nu_A(x)\) for all \(x, y \in X\).
Definition 4.3. An IFS $A = (\mu_A, \nu_A)$ in $X$ is called an intuitionistic L-fuzzy BP-ideal of $X$ if it satisfies:

1. $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$

2. $\mu_A(x) \geq \{\mu_A(x \ast y) \land \mu_A(y)\}$

3. $\nu_A(x) \leq \{\nu_A(x \ast y) \lor \nu_A(y)\}$ for all $x, y \in X$.

Example 4.4. Consider BP-algebra $X = \{0, 1, 2, 3\}$ in Example 2.3. Then $A = (\mu_A, \nu_A)$ defined as follows is an intuitionistic L-fuzzy BP-ideal of $X$:

$\mu_A(0) = \mu_A(2) = 1; \mu_A(1) = \mu_A(3) = t$; and $\nu_A(0) = \nu_A(2) = 0; \nu_A(1) = \nu_A(3) = s$

where $t, s \in [0, 1]$ and $t + s \leq 1$

Lemma 4.5. Let $A = (\mu_A, \nu_A)$ in $X$ be an intuitionistic L-fuzzy ideal of $X$.

If $x \ast y \leq z$ then:

$\mu_A(x) \geq \{\mu_A(x \ast y) \land \mu_A(y)\}$

$\nu_A(x) \leq \{\nu_A(y) \lor \nu_A(z)\}$

Proof: Let $x, y, z \in X$ such that $x \ast y \leq z$

Then $(x \ast y) \ast z = 0$ and thus

$$\mu_A(x) \geq \{\mu_A(x \ast y) \land \mu_A(y)\},$$

$$\geq \{\{\mu_A(x \ast y) \ast z\} \land \mu_A(z)\},$$

$$= \{\{\mu_A(0) \land \mu_A(z)\} \land \mu_A(y)\}$$

$$= \{\mu_A(y) \land \mu_A(z)\}$$

And similarly, $\nu_A(x) \leq \{\nu_A(y) \lor \nu_A(z)\}$

Lemma 4.6. Let $A = (\mu_A, \nu_A)$ in $X$ be an intuitionistic L-fuzzy BP-ideal of $X$.

If $x \leq y$ then:

$\mu_A(x) \geq \mu_A(y), \nu_A(x) \leq \nu_A(y)$ that is, $\mu_A$ is order reserving and $\nu_A$ is order-preserving.

Proof: Let $x, y, z \in X$ be such that $x \ast y = 0$

$$\mu_A(x) \geq \{\mu_A(x \ast y) \land \mu_A(y)\}$$

$$\geq \{\mu_A(0) \land \mu_A(y)\}$$

$$= \mu_A(y)$$
\[ \nu_A(x) \leq \{ \nu_A(x * y) \lor \nu_A(y) \} \]
\[ = \{ \nu_A(0) \lor \nu_A(y) \} \]
\[ = \nu_A(y) \]

**Definition 4.7.** A mapping \( f : X \to Y \) of BP-algebras is called a homomorphism if \( f(x * y) = f(x) * f(y) \) for all \( x, y \in X \). Let \( f : X \to Y \) be a homomorphism of BP-algebras for any IFS \( A = (\mu_A, \nu_A) \) in \( Y \), we define a new IFS by \( A^f = (\mu_A^f, \nu_A^f) \) in \( X \) by \( \mu_A^f(x) = \mu_A(f(x)) \), \( \nu_A^f(x) = \nu_A(f(x)) \) for all \( x, y \in X \).

**Theorem 4.8.** Let \( f : X \to Y \) be a homomorphism of BP-algebras. If an IFS \( A = (\mu_A, \nu_A) \) is an intuitionstic L-fuzzy BP-ideal of \( X \), then an IFS \( A^f = (\mu_A^f, \nu_A^f) \) in \( X \) is an intuitionstic L-fuzzy BP-ideal of \( X \).

**Proof** We have that \( \mu_A^f(x) = \mu_A(f(x)) \leq \mu_A(0) = \mu_A(f(0)) = \mu_A^f(0) \)
\( \nu_A^f(x) = \nu_A(f(x)) \geq \nu_A(0) = \nu_A(f(0)) = \nu_A^f(0) \) \( \forall x \in X \).

Let \( x, y \in X \). Then
\[ \{ \mu_A^f(x * y) \land \mu_A^f(y) \} = \{ \mu_A(f(x * y)) \land \mu_A(f(y)) \} \]
\[ = \{ \mu_A(f(x) * f(y)) \land \mu_A(f(y)) \} \]
\[ \leq \mu_A(f(x)) \]
\[ = \mu_A^f(x) \]
\[ \{ \nu_A^f(x * y) \lor \nu_A^f(y) \} = \{ \nu_A(f(x * y)) \lor \nu_A(f(y)) \} \]
\[ = \{ \nu_A(f(x) * f(y)) \lor \nu_A(f(y)) \} \]
\[ \geq \nu_A(f(x)) \]
\[ = \nu_A^f(x) \]

Hence \( A^f = (\mu_A^f, \nu_A^f) \) is an intuitionstic L-fuzzy BP-ideal of \( X \).

**Theorem 4.9.** Let \( f : X \to Y \) be an epimorphism of BP-algebra and let \( A = (\mu_A, \nu_A) \) be an IFS in \( Y \). If \( A = (\mu_A, \nu_A) \) is an intuitionstic L-fuzzy BP-ideal of \( Y \).

**Proof** For any \( x \in X \) there exist \( a \in X \) such that \( f(a) = x \).

Then,
\[ \mu_A(x) = \mu_A(f(a)) = \mu_A^f(a) \leq \mu_A^f(0) = \mu_A(f(0)) = \mu_A(0) \]
\[ \nu_A(x) = \nu_A(f(a)) = \nu_A^f(a) \geq \nu_A^f(0) = \nu_A(f(0)) = \nu_A(0) \]
Let $x, y \in X$. Then $f(a) = x$ and $f(b) = y$ for some $a, b \in X$.

Thus

$$\mu_A(x) = \mu_A(f(a)) = \mu_A^f(a)$$

$$\geq \{\mu_A^f(a \ast b) \land \mu_A(f(b))\}$$

$$= \{\mu_A(f(a \ast b)) \land \mu_A(f(b))\}$$

$$= \{\mu_A(f(a) \ast f(b)) \land \mu_A(f(b))\}$$

$$= \{\mu_A(x \ast y) \land \mu_A(y)\}$$

$$\nu_A(x) = \nu_A(f(a)) = \nu_A^f(a)$$

$$\leq \{\nu_A^f(a \ast b) \lor \nu_A(f(b))\}$$

$$= \{\nu_A(f(a \ast b)) \lor \nu_A(f(b))\}$$

$$= \{\nu_A(f(a) \ast f(b)) \lor \nu_A(f(b))\}$$

$$= \{\nu_A(x \ast y) \lor \nu_A(y)\}$$

References


