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GEOMETRIC MEAN CORDIAL LABELING OF SUBDIVISION OF STANDARD GRAPHS

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Abstract: Let $G = (V, E)$ be a graph and f be a mapping from $V(G) \to \{0, 1, 2\}$. For each edge uv, assign the label $\left\lceil \sqrt{f(u)f(v)} \right\rceil$, f is called a *geometric mean cordial labeling* if $|\ v_f(i) - v_f(j)\ | \le 1$ and $|\ e_f(i) - e_f(j)\ | \le 1$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges labeled with x, $x \in \{0, 1, 2\}$ respectively. A graph with a geometric mean cordial labeling is called *geometric mean cordial graph*.

In this paper, the geometric mean cordiality of subdivision of path, cycle, star and complete bipartite graphs are discussed.

AMS Subject Classification: 05C78

Key Words: cordial labeling, cordial graphs, geometric mean cordial labeling, geometric mean cordial graphs, subdivision

1. Introduction

In this paper, we use the definition of subdivision and apply it to standard graphs such as path, cycle, star and complete bipartite graphs. Here we give results, definitions, theorems, examples, table and explanations mainly on geometric mean labeling of subdivision of standard graphs. Also we find whether the subdivision of standard graphs obeys geometric mean cordial or not.

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In many cases in graph theory (and mathematics in general), elementary subdivision (sometimes known as an expansion)[7] is used to compare graphs for similarities by subdividing each edge e_i many times in labeled graphs and subdivision may refer to a portion of a country or other political division established for the purpose of the government by communicating through internal vertices of degree 2 into edges. Also a grid map uses a uniform subdivision of the world into small regular shapes.

Definition 1.1. [6] $\{0, 1, 2\}$. For each edge uv, assign the label

$$\left\lceil \sqrt{f(u)f(v)} \right\rceil$$

, f is called a *geometric mean cordial labeling* if $\mid v_f(i) - v_f(j) \mid \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges labeled with x, $x \in \{0, 1, 2\}$ respectively. A graph which admits a geometric mean cordial labeling is called *geometric mean cordial graph.*

In the paper [6], we have proved the some standard graphs which are geometric mean cordial and some graphs which are not geometric mean cordial.

2. Main Results

Motivated by the concept of geometric mean cordial labeling, we give theorems based on geometric mean cordial labeling of subdivision of standard graphs as follows.

Definition 2.1. [1] A subdivision of a graph G denoted by $S(G)$ (sometimes known as an expansion) is a graph resulting from the subdivision of edges in G. The subdivision of some edge e with endpoints u, v yields a graph containing one new vertex w, and with an edge set replacing e by two new edges, uw and wv.

We illustrate the definition by the following example.

Example 2.2. Consider the following graph

In this example, the edge e is subdivided into two new edges uw and wv.

Now we will check the geometric mean cordiality of subdivision of some standard graphs.

Results 2.3. *Since $S(P_n)$ is also P_{2n-1} which is also a path and already proved that path is geometric mean cordial [6].*

Results 2.4. *and $S(C_n)$ is also C_{2n} which is also a cycle and already proved that cycle is geometric mean cordial iff $n \equiv 1, 2 (mod\ 3)$.[6]*

Now we will prove subdivision of Star graph is geometric mean cordial.

Theorem 2.5. *$S(K_{1,n})$ is geometric mean cordial.*

Proof. Let u be the central vertex and $v_1, v_2, \ldots, v_n$ be the pendant vertices of $K_{1,n}$. Also, let $u_1, u_2, \ldots, u_n$ be the subdivisional vertices of $S(K_{1,n})$. Now $S(K_{1,n})$ has $2n+1$ vertices and $2n$ edges.

Case(i): $n \equiv 0(mod 3)$. Let $n = 3t$.
Define $f(u) = 1$,

$$f(u_i) = 0, \quad 1 \leq i \leq t, \qquad f(v_i) = 0, \quad 1 \leq i \leq t,$$
$$f(u_{t+i}) = 1, \quad 1 \leq i \leq 2t, \qquad f(v_{t+i}) = 2, \quad 1 \leq i \leq 2t.$$

By the definition of f, we see that $v_f(0) = v_f(2) = 2t, v_f(1) = 2t+1$.

The central vertex u which is labeled as 1 is adjacent to t subdivisional vertices $u_1, u_2, \ldots, u_t$ which are labeled with 0 and these t vertices are adjacent to t end vertices $v_1, v_2, \ldots, v_t$ respectively which are labeled with 0 and so $e_f(0) = 2t$.

Also the central vertex u which is labeled as 1 is adjacent to $2t$ subdivisional vertices $u_{t+1}, u_{t+2}, \ldots, u_{3t}$ which are labeled with 1 and so $e_f(1) = 2t$.
Now these $2t$ vertices are adjacent to $2t$ end vertices $v_{t+1}, v_{t+2}, \ldots, v_{3t}$ respectively which are labeled with 2 and so $e_f(2) = 2t$.
Therefore $e_f(0) = e_f(1) = e_f(2) = 2t$.

Case(ii): $n \equiv 1 (mod 3)$. Let $n = 3t + 1$.

Assign the labels to the vertices u, u_i, v_i according to case(i), $1 \leq i \leq n-1$. Then assign the label 2 to the vertex u_n and 0 to the vertex v_n.

By the definition of f, we see that $v_f(0) = v_f(1) = v_f(2) = 2t + 1$. By the similar argument as in case (i) we have $2t$ edges contribute to each $e_f(0)$, $e_f(1)$ and $e_f(2)$.

Now the central vertex u labeled with 1 is adjacent to the remaining only one subdivisional vertex which is labeled 2, also the same subdivisional

vertex which is labeled 2 is adjacent to only one pendant vertex which is labeled 0. Then 1 edge contributes to each $e_f(0)$ and $e_f(2)$. Thus $e_f(0) = e_f(2) = 2t + 1, e_f(1) = 2t$.

Case(iii): $n \equiv 2 (mod 3)$. Let $n = 3t + 2$.

Assign the labels to the vertices u, u_i, v_i according to case(ii), $1 \leq i \leq n-1$. Then assign the label 1 to the vertex u_n and 2 to the vertex v_n.

By the definition of f, we see that $v_f(0) = 2t + 1, v_f(1) = v_f(2) = 2t + 2$. By the similar argument as in case (ii) we have $2t + 1$ edges contribute to each $e_f(0)$, $e_f(2)$ and $2t$ edges contribute to $e_f(1)$.

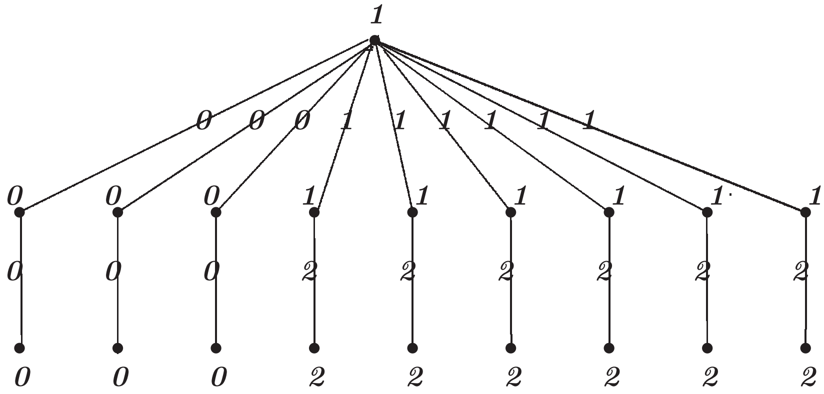
Now the subdivisional vertex which is labeled 1 is adjacent to u which is labeled 1 and only one pendant vertex which is labeled 2. Then it contributes 1 edge to $e_f(1)$ and 1 edge to $e_f(2)$.
Thus $e_f(0) = e_f(1) = 2t + 1, e_f(2) = 2t + 2$.

From all the three cases above, we see that $| v_f(i) - v_f(j)| \leq 1$ and $| e_f(i) - e_f(j) | \leq 1$, for all $i, j \in \{0, 1, 2\}$ and hence f is a geometric mean cordial labeling. □

The geometric mean cordial labeling of $S(K_{1,n})$ is illustrated in the following example.

Example 2.6.

Case(i): *Consider the graph* $S(K_{1,9})$.



Here $v_f(0) = v_f(2) = 6, v_f(1) = 7$ *and* $e_f(0) = e_f(1) = e_f(2) = 6.$

Theorem 2.7. *$S(K_{2,n})$ is not geometric mean cordial.*

Proof. Let $V(K_{2,n}) = V_1 \bigcup V_2$, where $V_1 = \{u, v\}$ and $V_2 = \{w_1, w_2, ..., w_n\}$ and $u_1, u_2, ..., u_n$ be the subdivisional vertices for the edges $uw_1, uw_2, ..., uw_n$ and $v_1, v_2, ..., v_n$ be the subdivisional vertices for the edges $vw_1, vw_2, ..., vw_n$. Then $S(K_{2,n})$ has $3n + 2$ vertices and $4n$ edges.

Case(i): $n \equiv 0(mod3)$. Let $n = 3t$.
Then $S(K_{2,n})$ has $9t + 2$ and $12t$ edges.

Suppose $S(K_{2,n})$ admits a geometric mean cordial labeling. Then it should have the three possibilities.
(i) $v_f(0) = v_f(1) = 3t + 1$, $v_f(2) = 3t$,
(i) $v_f(0) = v_f(2) = 3t + 1$, $v_f(1) = 3t$,
$(iii) v_f(1) = v_f(2) = 3t + 1$, $v_f(0) = 3t$,
and $e_f(0) = e_f(1) = e_f(2) = 4t. \longrightarrow (1)$

Subcase(i): Suppose $f(u) = f(v) = 0$.

Without loss of generality, label the first t vertices $w_1, w_2, ..., w_t$ by 0, next t vertices $w_{t+1}, w_{t+2}, ..., w_{2t}$ by 1, and last t vertices $w_{2t+1}, w_{2t+2}, ..., w_{3t}$ by 2.

Also label the subdivisional vertices $u_i (1 \leq i \leq n)$ by the labels as in $w_i (1 \leq i \leq n)$ respectively and similarly for the vertices $v_i (1 \leq i \leq n)$. Then the edges of $S(K_{2,n})$ will get the following labels.

Edges	i	$e_f(0)$	$e_f(1)$	$e_f(2)$
uu_i	$1 \leq i \leq 3t$	$3t$	-	-
$u_i wi$	$1 \leq i \leq t$	t	-	-
$u_i wi$	$t+1 \leq i \leq 2t$	-	t	-
$u_i wi$	$2t+1 \leq i \leq 3t$	-	-	t
vv_i	$1 \leq i \leq 3t$	$3t$	-	-
$v_i wi$	$1 \leq i \leq t$	t	-	-
$v_i wi$	$t+1 \leq i \leq 2t$	-	t	-
$v_i wi$	$2t+1 \leq i \leq 3t$	-	-	t
	Total	$8t$	$2t$	$2t$

From the above case, we see that $e_f(0) = 8t, e_f(1) = e_f(2) = 2t$, which is a contradiction to (1). Hence f is not a geometric mean cordial labeling.

Subcase(ii): $f(u) = 0, f(v) = 1$.
Then we get $e_f(0) = 6t, e_f(1) = e_f(2) = 3t$.
Subcase(iii): $f(u) = 1,\ f(v) = 0$.
Similar as in Subcase (ii).
Subcase(iv): $f(u) = 0, f(v) = 2$.
Then we get $e_f(0) = 6t$, $e_f(1) = 2t$, $e_f(2) = 4t$.
Subcase(v): $f(u) = 2, f(v) = 0$.
Similar as in Subcase (iv).
Subcase(vi): $f(u) = 1, f(v) = 1$.
Then we get $e_f(0) = e_f(1) = e_f(2) = 4t$.
Subcase(vii): $f(u) = 1, f(v) = 2$.
Then we get $e_f(0) = 4t, e_f(1) = 3t, e_f(2) = 5t$.
Subcase(viii): $f(u) = 2, f(v) = 1$.
Similar as in Subcase (vii).
Subcase(ii): $f(u) = f(v) = 2$.
Thus we get $e_f(0) = 4t, e_f(1) = 2t, e_f(2) = 6t$.

In all the above subcases, we observe that if either or both of the labeling of vertices u or v are zero, then $e_f(0) \leq 4t$. Also if either or both of the vertices u and v having the label 2, then $e_f(2) \leq 4t$. So in all the subcases, we see that $S(K_{2,n})$ is not a geometric mean cordial.

Thus in the following cases, we consider only the subcase of labeling of both the vertices u and v are 1.

Case(ii): n $\equiv 1(mod3)$. Let $n = 3t + 1$.
Then $S(K_{2,n})$ has $9t + 5$ and $12t + 4$ edges.

Suppose $S(K_{2,n})$ admits a geometric mean cordial labeling. Then it should

have the three possibilities.
(i) $v_f(0) = v_f(1) = 3t + 2$, $v_f(2) = 3t + 1$,
(i) $v_f(0) = v_f(2) = 3t + 2$, $v_f(1) = 3t + 1$,
$(iii) v_f(1) = v_f(2) = 3t + 2$, $v_f(0) = 3t + 1$, and also
(i) $e_f(0) = e_f(1) = 4t + 2$, $e_f(2) = 4t$,
(ii) $e_f(0) = e_f(2) = 4t + 2$, $e_f(1) = 4t$,
$(iii) e_f(1) = e_f(2) = 4t + 2$, $e_f(0) = 4t$. $\longrightarrow$(2)

Suppose $f(u) = f(v) = 1$.

Assign the labels as in case (i) upto $3t$ vertices. Let the last vertices u_{3t+1}, v_{3t+1} and w_{3t+1} be assigned 0 or 1 or 2. Then the edges of $S(K_{2,n})$ will get the following labels. The combinations are followed in the table as follows:

Subcases	u_{3t+1}	v_{3t+1}	w_{3t+1}
(a)	1	2	0
(b)	2	1	0
(c)	0	2	1
(d)	2	0	1
(e)	0	1	2
(f)	1	0	2

For the above combinations, the edges uu_{3t+1}, $u_{3t+1}w_{3t+1}$, vv_{3t+1}, v_{3t+1} w_{3t+1} contribute to $e_f(0)$, $e_f(1)$ and $e_f(2)$ as follows.

	(uu_{3t+1})			$(u_{3t+1}w_{3t+1})$			(vv_{3t+1})			$(v_{3t+1}w_{3t+1})$	
0	1	2	0	1	2	0	1	2	0	1	2
-	1	-	1	-	-	-	-	1	1	-	-
-	-	1	1	-	-	-	1	-	1	-	-
1	-	-	1	-	-	-	-	1	-	-	1
-	-	1	-	-	1	1	-	-	1	-	-
1	-	-	1	-	-	-	1	-	-	-	1
-	1	-	-	-	1	1	-	-	1	-	-

Finally we get $e_f(0)$, $e_f(1)$ and $e_f(2)$ in 6 Subcases as follows:

$e_f(0)$	$e_f(1)$	$e_f(2)$
2	1	1
2	1	1
2	0	2
2	0	2
2	1	1
2	1	1

From the above table, in subcases (a) and (b), $e_f(0) = 2, e_f(1) = e_f(2) = 1$.
We know that in subcase (vi) of Case (i), $e_f(0) = e_f(1) = e_f(2) = 4t$.
Totally we get $e_f(0) = 4t + 2, e_f(1) = e_f(2) = 4t + 1$. But in this case, we have $v_f(0) = v_f(2) = 3t + 1, v_f(1) = 3t + 3$, a contradiction to

$$v_f(0) = v_f(1) = 3t + 2, \quad v_f(2) = 3t + 1.$$

By the similar argument as in subcases (a) and (b), we see that in subcase (c) and (d), we get $e_f(0) = e_f(2) = 4t + 2, e_f(1) = 4t$, a contradiction to (2). In subcases (e) and (f), we see that it is similar to subcase (a) and (b). In all the above cases, we see that $S(K_{2,n})$ is not a geometric mean cordial.

Case(iii): n $\equiv 2(mod\ 3)$. Let $n = 3t + 2$.
Then $S(K_{2,n})$ has $9t + 8$ vertices and $12t + 8$ edges. Now we have three possibilities.
(i) $v_f(0) = v_f(1) = 3t + 3$, $v_f(2) = 3t + 2$,
(i) $v_f(0) = v_f(2) = 3t + 3$, $v_f(1) = 3t + 2$,
$(iii) v_f(1) = v_f(2) = 3t + 3$, $v_f(0) = 3t + 2$, and also
(i) $e_f(0) = e_f(1) = 4t + 3$, $e_f(2) = 4t + 2$,
(i) $e_f(0) = e_f(2) = 4t + 3$, $e_f(1) = 4t + 2$,
$(iii) e_f(1) = e_f(2) = 4t + 3$, $e_f(0) = 4t + 2$,

Assign the labels as in case (i) upto $3t$ vertices. Let the last vertices w_{3t+1}, w_{3t+2}, u_{3t+1}, u_{3t+2}, v_{3t+1}, v_{3t+2} be asssigned 0 or 1 or 2. Then the edges of $S(K_{2,n})$ will get the following labels and 18 combinations.

Subcases	w_{3t+1}	w_{3t+2}	u_{3t+1}	u_{3t+2}	v_{3t+1}	v_{3t+2}
(a)	0	0	1	1	2	2
(b)	0	0	2	2	1	1
(c)	0	1	1	0	2	2
(d)	0	1	2	2	1	0
(e)	1	0	0	1	2	2
(f)	1	0	2	2	0	1
(g)	0	2	1	0	2	1
(h)	0	2	2	1	1	0
(i)	2	0	0	1	1	2
(j)	2	0	1	2	0	1
(k)	1	1	0	0	2	2
(l)	1	1	2	2	0	0
(m)	1	2	0	0	2	1
(n)	1	2	2	1	0	0
(o)	2	1	0	0	1	2
(p)	2	1	1	2	0	0
(q)	2	2	0	0	1	1
(r)	2	2	1	1	0	0

For the above combinations, the edges uu_{3t+1}, uu_{3t+2}, vv_{3t+1}, vv_{3t+2}, $u_{3t+1}w_{3t+1}$, $v_{3t+1}w_{3t+1}$, $u_{3t+2}w_{3t+2}$, $v_{3t+2}w_{3t+2}$, respectively in 1 to 8 columns in order contribute $e_f(0)$, $e_f(1)$ and $e_f(2)$ and we have total of $e_f(0)$, $e_f(1)$ and $e_f(2)$ in the last column..

Subcases	0 1 2	0 1 2	0 1 2	0 1 2	0 1 2	0 1 2	0 1 2	0 1 2	$e_f(0)$ $e_f(1)$ $e_f(2)$
(a)	- 1 -	- 1 -	- - 1	- - 1	1 - -	1 - -	1 - -	1 - -	4 2 2
(b)	- - 1	- - 1	- 1 -	- 1 -	1 - -	1 - -	1 - -	1 - -	4 2 2
(c)	- 1 -	1 - -	- - 1	- - 1	1 - -	1 - -	1 - -	- - 1	4 1 3
(d)	- - 1	- - 1	- 1 -	1 - -	1 - -	1 - -	- - 1	1 - -	4 1 3
(e)		*Same*	*as*	*Subcase*	(c)				
(f)		*Same*	*as*	*Subcase*	(d)				
(g)	- 1 -	1 - -	- - 1	- 1 -	1 - -	1 - -	1 - -	- - 1	4 2 2
(h)	- - 1	- 1 -	- 1 -	1 - -	1 - -	1 - -	- - 1	1 - -	4 2 2
(i)		*Same*	*as*	*Subcase*	(g)				
(j)		*Same*	*as*	*Subcase*	(h)				
(k)	1 - -	1 - -	- - 1	- - 1	1 - -	- - 1	1 - -	- - 1	4 - 4
(l)	- - 1	- - 1	1 - -	1 - -	- - 1	1 - -	- - 1	1 - -	4 - 4
(m)	1 - -	1 - -	- - 1	- 1 -	1 - -	- - 1	1 - -	- - 1	4 1 3
(n)	- - 1	- 1 -	1 - -	1 - -	- - 1	1 - -	- - 1	1 - -	4 1 3
(o)		*Same*	*as*	*Subcase*	(m)				
(p)		*Same*	*as*	*Subcase*	(n)				
(q)	1 - -	1 - -	- 1 -	- 1 -	1 - -	- - 1	1 - -	- - 1	4 2 2
(r)	- 1 -	- 1 -	1 - -	1 - -	- - 1	1 - -	- - 1	1 - -	4 2 2

We know that in subcase (vi) of case (i), $e_f(0) = e_f(1) = e_f(2) = 4t$.
From the above table, in subcases (a), (b), (g), (h), (i), (j), (q), (r), we see that $e_f(0) = 4, e_f(1) = e_f(2) = 2$. Finally we get $e_f(0) = 4t + 4, e_f(1) = e_f(2) = 4t + 2$.

Also in subcases (c), (d), (e), (f), (m), (n), (o), (p), we see that $e_f(0) = 4, e_f(1) = 1, e_f(2) = 3$. Finally we get $e_f(0) = 4t + 4, e_f(1) = 4t + 1, e_f(1) = 4t + 3$.

Finally in subcases (k) and (l) , we that $e_f(0) = e_f(2) = 4, e_f(1) = 0$. Thus we get $e_f(0) = e_f(2) = 4t + 4, e_f(1) = 4t$. Therefore in all the above subcases, we see that f is not a geometric mean cordial labeling.

□

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