Wiener dimension of spiders, $k$-ary trees and Binomial trees

D. Azubha Jemilet$^1$ and Indra Rajasingh$^{1,*}$

$^1$School of Advanced Sciences, VIT University, Chennai, India
azujemi@gmail.com

Abstract

The distance of a vertex $v$ in a graph $G$ is the sum of the distances between $v$ and every other vertex in $G$. The Wiener dimension of a connected graph is defined as the number of different distances of its vertices. In this paper we compute the Wiener dimension of spiders, $k$-ary trees and Binomial trees.

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1 Introduction

Topological indices are numerical parameters of a graph which characterize its topology and are usually graph invariants. Topological indices are used in the development of quantitative structure-activity relationships (QSARs) in which the biological activity or other properties of molecules are correlated with their chemical structures. The Wiener index introduced in 1947 is the oldest topological index related to molecular branching [1]. Based on its success, many other topological indexes of chemical graphs have been developed subsequent to Wiener’s work. The Wiener index is also closely related to the closeness centrality of a vertex in a graph, a quantity inversely proportional to the sum of all distances between the given vertex and all other vertices and has been frequently used in sociometry and the theory of social networks [2]. In a graph $G$, the distance $d_G(u,v)$ between two vertices $u, v$ in $G$ is the minimum number of edges on a path in $G$ between $u$ and $v$. Wiener index $W(G)$ of a graph $G$ is defined as the

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sum of distances between all pairs of vertices in $G$. In other words $W(G) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v)$. Wiener index is extensively studied in Chemistry [3, 4] and Mathematics [1, 5, 6]. For a vertex $u$ in graph $G$, the distance or closeness of $u$ is defined as $\delta_G(u) = \sum_{v \in V(G)} d_G(u,v)$. Suppose that $\{\delta_G(u) | u \in V(G)\} = \{\delta_1, \delta_2, \ldots, \delta_k\}$, then the Wiener dimension $\text{dim}_W(G)$ is defined as $k$. Alizadeh et al. [7] have studied Wiener dimension of $(5,0)$-Nanotubical Fullerenes.

## 2 Preliminaries

Wiener index can also be defined as $W(G) = \frac{1}{2} \sum_{u \in V(G)} \delta_G(u)$ where $\frac{1}{2}$ compensates for the fact that each path between $u$ and $v$ is counted in $\delta_G(u)$ as well as in $\delta_G(v)$. When there is no ambiguity, we denote $\delta_G(u)$ as $\delta(u)$. A tree is an undirected acyclic graph in which any two vertices are connected by exactly one path [8].

**Definition 2.1.** Let $T$ be a tree. A vertex $v \in V(T)$ is called branching point of $T$ if $\text{deg}_T(v) \geq 3$. If $\text{deg}_T(v) = 1$, the vertex $v$ is named a pendant vertex or a leaf of $T$.

**Definition 2.2.** Let $v \in V(T)$. For $e \in E(T)$, define congestion on $e$ with respect to $v$, denoted by $c_v(e)$, as the number of times $e$ is crossed while traversing from $v$ to every other vertex of $T$.

**Theorem 2.3.** [9] Let $v \in V(T)$. Then $\delta_T(v) = \sum_{e \in E(T)} c_v(e)$.

**Lemma 2.4.** (T-Closeness Lemma) [9] Let $T$ be a tree and $v \in V(T)$. For every edge $e$ in $T$, let $T_e$ be the component of $T - e$ which does not contain $v$. Then $\delta(v) = \sum_{e \in E(T)} |V(T_e)|$.

**Definition 2.5.** Let $v$ be a cut vertex of $G$. The $v$-components of $G$ are subgraphs induced by the components of $G - v$ together with $v$.

Jemilet et al. [9] have determined the Wiener dimension of a crystal tree as 3, firecracker graph as 3 and a banana tree as 4.

In this paper we determine the Wiener dimension of spider trees, complete $k$-ary trees and binomial trees.

## 3 Wiener dimension of spider trees and $k$-ary trees

**Definition 3.1.** [10] Let $T^*$ be the tree formed from a star $K_{1,n}$ by subdividing any number of its edges any number of times; that is, $T^*$
has at most one vertex of degree 3 or more. We call such a tree $T^*$ as spider tree. See Figure 1(a) and 1(b). The vertex of degree $n$ in $K_{1,n}$ is said to be at level 1 in the spider tree. Their descendents are in level 2 and so on. If each edge of $K_{1,n}$ is subdivided equal number of times, then the spider tree is called a uniform spider tree.

**Theorem 3.2.** Wiener dimension of a uniform spider of level $k$ is $k$.

*Proof.* By symmetry, $\delta(v)$ for all vertices $v$ in the same level are equal. Let $v_1$ and $v_2$ be vertices at different levels in a path with the root $v$ as one end and a pendant vertex $w$ at the other end. Let $e$ be an edge between $v_1$ and its parent node. Then $T - e = T_1^e \cup T_2^e$ where $T_1^e$ is a subtree not containing $v_1$ and $v_2$ and $T_2^e$ is a subtree containing $v_1$ and $v_2$. The contributions of vertices in $V(T_i^e)$ to $c_{v_1}(e)$ and $c_{v_2}(e)$ are the same. In the same manner, let $f$ be an edge on the path between $v_2$ and $w$. Let $d(v_2, w) = s$. Then $T - f = T_1^f \cup T_2^f$ where $T_1^f$ contains $v_1$ and $v_2$ and $T_2^f$ does not contain $v_1$ and $v_2$. Hence the contributions of vertices in $V(T_i^f)$ to $c_{v_1}(f)$ and $c_{v_2}(f)$ are the same. Therefore to compute $\delta(v_2) - \delta(v_1)$, we need to consider the congestion on the edges in the path between $v_1$ and $v_2$. Let $v_1 e_1 u_1 e_2 u_2 \ldots u_{t-1} e_t v_2$ be the path between $v_1$ and $v_2$. Then $\delta(v_2) - \delta(v_1) = [(t-1) + (t-2) + \ldots + 1] + (t-1)(t+s) - [(t-1) + (t-2) + \ldots + 1] + (t-1)s = (t-1)t > 0$. Therefore $\delta(v_2) \neq \delta(v_1)$. Therefore $\dim_w(T^*) = k$. See Figure 2. 

**Definition 3.3.** [11, 12] A complete binary tree of height $h$ is a binary tree which contains exactly $2^r$ nodes at height $r$, $0 \leq r \leq h$. In this tree, every node at depth less than $h$ has two children. The nodes at height $h$ are the leaves. Complete binary tree is denoted as $T_h^k$ and has $2^h$ nodes. A $k$-ary tree is a rooted tree in which each node has no more than $k$ children. It is also known as a $k$-way tree.
Theorem 3.4. Let $T^* \kappa$ be a $\kappa$-ary tree of height $r$. Then $\operatorname{dim}_W(T^* \kappa) = r$ for $r \geq 1$.

The proof is similar to that of Theorem 3.2.

4 Trees with Wiener dimension $2^n$

Definition 4.1. [13] A binomial tree $B_0$ of height 0 is a single vertex. For all $n \geq 0$, a binomial tree $B_n$ of height $n$ is a tree formed by joining the roots of two binomial trees of height $n - 1$ with a new edge and designating one of these roots to be the root of the new tree. A binomial tree of height $n$ has $2^n$ vertices.

Theorem 4.2. Let $T$ be a binomial tree $B_n$. Then $\operatorname{dim}_W(B_n) = 2^n$.

Proof. We prove the result by induction on $n$. The result is trivial for $n = 1$. See Figure 3(a). Assume the result to be true for $B_{n-1}$. Let $e$ denote the edge joining the roots of two copies of $B_{n-1}$. By induction hypothesis, the distance of any two corresponding vertices in the two
copies of $B_{n-1}$ are the same. The distance of vertex $v$ namely $\delta(v)$ in $B_n$ now depends on the cut vertices $u$ and $v$ where $e = (u, v)$ connecting the two roots of binary trees in $B_{n-1}$. Hence, again by symmetry, the distances of vertices increases uniformly. Hence $\dim_w(B_n) = 2^n$. See Figure 3(a) and 3(b).

\section{Conclusion}

We have proved that Wiener dimension is $k$ for spider trees of level $k$ and $r$ for complete $k$-ary trees of height $r$. For binomial trees of dimension $n$, Wiener dimension is $2^n$. It is an interesting line of research to characterize trees with Wiener dimension $k$, $k \geq 3$. It would also be interesting to develop strategies to compute Wiener dimension of general graphs.

\section*{References}


