POWER DOMINATION PARAMETERS IN HYPERMESH-PYRAMID NETWORKS AND CORONA GRAPHS

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Abstract: A set $S$ is said to be a $k$-power dominating set of $G$ if every vertex in the system is monitored by the set $S$ following a set of rules for power system monitoring. A zero forcing set of $G$ is a subset of vertices $B$ such that if the vertices in $B$ are colored blue and the remaining vertices are colored white initially, repeated application of the color change rule can color all vertices of $G$ blue. The $k$-power domination number and the zero forcing number of $G$ are the minimum cardinality of a $k$-power dominating set and the minimum cardinality of a zero forcing set respectively of $G$.

In this paper, we introduce two new parameters namely total $k$-power domination number and connected $k$-power domination number and find $k$-power domination number and zero forcing number for hypermesh network, hypermesh-pyramid network and corona graph.

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Key Words: $k$-power dominating set; total $k$-power dominating set; connected $k$-power dominating set; zero-forcing set; hypermesh network; pyramid-hypermesh; corona graph.
1. Introduction

We begin with the following definitions.

**Definition 1.1.** [1] For $v \in V(G)$, the open neighbourhood of $v$, denoted as $N_G(v)$, is the set of vertices adjacent with $v$; and the closed neighbourhood of $v$, denoted by $N_G[v]$, is $N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the open neighbourhood of $S$ is defined as $N_G(S) = \bigcup_{v \in S} N_G(v)$ and the closed neighbourhood of $S$ is defined as $N_G[S] = N_G(S) \cup S$. For brevity, we denote $N_G(S)$ and $N_G[S]$ by $N(S)$ and $N[S]$ respectively.

**Definition 1.2.** [1] For a graph $G(V, E)$, $S \subseteq V$ is a dominating set of $G$ if every vertex in $V \setminus S$ has at least one neighbour in $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$.

**Definition 1.3.** [1] Let $G(V, E)$ be a graph and let $S \subseteq V(G)$. For $k \geq 0$, we define the sets $M^i(S)$ of vertices monitored by $S$ at level $i$, $i \geq 0$, inductively as follows:
1. $M^0(S) = N[S]$.
2. $M^{i+1}(S) = \bigcup \{N[v] : v \in M^i(S) \text{ such that } |N[v] \setminus M^i(S)| \leq k\}$.

If $M^\infty(S) = V(G)$, then the set $S$ is called a $k$-power dominating set of $G$. The minimum cardinality of a $k$-power dominating set in $G$ is called the $k$-power domination number of $G$ written $\gamma_{p,k}(G)$.

In general, the $k$-power domination problem is NP-complete [1]. In fact, the problem has been shown to be NP-complete even when restricted to bipartite graphs and chordal graphs [1].

**Definition 1.4.** Color change rule: [15] Let $G$ be a graph with each vertex colored either white or blue. If $u$ is a blue vertex and exactly one neighbour $w$ of $u$ is white, then change the color of $w$ to blue. We say that $u$ forces $w$ and denote it by $u \rightarrow w$.

**Definition 1.5.** [15] A zero forcing set of $G$ is a subset of vertices $B$ such that when the vertices in $B$ are colored blue and the remaining vertices are colored white initially, repeated application of the color change rule can color all vertices of $G$ blue. The zero forcing number, denoted $Z(G)$ of $G$ is the minimum cardinality of a zero forcing set of $G$.

In line with the various domination parameters, we introduce the following definitions with respect to $k$-power dominating sets.

**Definition 1.6.** A $k$-power dominating set $S$ is called a total $k$-power dominating set if $S$ contains no isolated vertex. The total $k$ power domination
number of $G$, denoted $\gamma_{p,k}^{t}(G)$ is the minimum cardinality of a total $k$-power dominating set of $G$.

**Definition 1.7.** A $k$-power dominating set $S$ is called a connected $k$-power dominating set if the subgraph induced by $S$ is connected. The connected $k$-power domination number of $G$, denoted $\gamma_{p,k}^{c}(G)$ is the minimum cardinality of a connected $k$-power dominating set of $G$.

In this paper, we solve $k$-power domination, total $k$-power domination, connected $k$-power domination and zero forcing problems for hypermesh network, pyramid-hypermesh network and corona graphs.

![Figure 1](image_url)

Figure 1: Circled vertices indicate power dominating set of $G$ (a) Hypermesh $HM_{4,4}$ (b) Hypermesh Pyramid $P_{HM,2}$

### 1.1. Hypermesh Network

**Definition 1.8.** [17] An $m \times n$ hypermesh, $HM_{m,n}$, is a set of nodes $V(HM_{m,n}) = \{(x,y) \mid 1 \leq x \leq m, 1 \leq y \leq n\}$ where nodes $(x_1, y_1)$ and $(x_2, y_2)$ are connected by an edge if and only if $x_1 = x_2$ or $y_1 = y_2$. See Figure 1(a).

**Theorem 1.9.** Let $G$ be the hypermesh $HM_{m,n}$, $m, n \geq k + 2, m \leq n$. Then $\gamma_{p,k}(G) \geq m - k$.

**Proof.** Suppose $S$ is a $k$-power dominating set of $HM_{m,n}$ with $|S| = m - k - 1$. Every member of $S$ dominates every other member of the row and column to which it belongs. Then $N[S]$ induces a subgraph $H$ with $x$ rows and $y$ columns
where \(x, y \leq m - k - 1\). Therefore, \(HM_{m,n} \setminus H\) is \(HM_{\alpha,\beta}\) with \(\alpha \geq k + 1\) and \(\beta \geq k + 1\). Choose \(u \in N[S] \setminus S\). Then \(u\) is adjacent to \(k + 1\) vertices in \(HM_{\alpha,\beta}\), a contradiction. Hence \(\gamma_{p,k}(G) \geq m - k\). See Figure 2.

**Proof of Correctness:** \(M^0(S) = N[S]\) contains all vertices in the first column and all vertices in the first \(m - k\) rows of \(G\). Each vertex in \(M^0(S)\) other than the first column is adjacent to \(k\) vertices in \(V(G) \setminus N[S]\). Hence \(M^1(S) = V(G)\). Since the subgraph induced by \(S\) is connected, we have \(\gamma_{p,k}(G) = \gamma_{p,k}^c(G) = m - k\). See Figure 1(a).

**Theorem 1.10.** Let \(G\) be the hypermesh \(HM_{m,n}\), \(m, n \geq k + 2\), \(m \leq n\). Then \(\gamma_{p,k}(G) = \gamma_{p,k}^t(G) = \gamma_{p,k}^c(G) = m - k\).

### 2. Hypermesh-Pyramid Network

**Definition 2.1.** [17] A hypermesh-pyramid of \(n\) levels, denoted \(P_{HM,n}\), consists of a set of vertices \(V(P_{HM,n}) = \{(r, x, y) \mid 0 \leq r \leq n, 1 \leq x, y \leq 2^r\}\). A vertex \((r, x, y) \in V(P_{HM,n})\) is said to be a vertex at level \(r\). All the vertices in
level \( r \) form a \( 2^r \times 2^r \) hypermesh network. In \( P_{HM,n} \), there are \( N = \sum_{r=0}^{n} 4^r = (4^{n+1}) - 1)/3 \) vertices. We refer to adjacent vertices at the same level as sister vertices. Every vertex \((r, x, y)\) in level \( r \) is adjacent to exactly 4 vertices, namely \( (r+1, 2x-1, 2y), (r+1, 2x, 2y-1), (r+1, 2x-1, 2y-1) \) and \( (r+1, 2x, 2y) \), at level \( r+1 \). Further \((r, x, y)\) is adjacent to exactly one vertex, namely \((r-1, \left\lfloor \frac{x}{2} \right\rfloor, \left\lfloor \frac{y}{2} \right\rfloor)\) in level \( r-1 \) as its father vertex. The apex vertex in \( P_{HM,n} \) is the vertex with address \((0, 1, 1)\). It is adjacent to \((1,0,0), (1,0,1), (1,1,0)\) and \((1,1,1)\), which are its children. So the degree of the apex is 4. The corner vertices of \( P_{HM,n} \) are vertices \((n,1,1), (n,2^n,1), (n,2^n,2^n)\) and \((n,1,2^n)\) respectively. See Figure 1(b).

### 2.1. \( k \)-power domination in Hypermesh Pyramid Network

Let \( G \) be the pyramid network \( P_{HM,n} \). In each level \( r, \ 0 \leq r \leq n \), there are \( 4^r \) vertices inducing \( 4^r \) hypermesh network \( HM_m,n \). By definition, 4 vertices in level \( r, \ 0 \leq r \leq n \), are adjcent to exactly one vertex in level \( r-1 \). Hence it is enough to consider level \( n \) of \( P_{HM,n} \) to determining the \( k \)-power dominating set or the zero forcing set of \( P_{HM,n} \). Choosing the \( k \)-power dominating set or zero forcing set in level \( n \) is the minimum \( k \)-power dominating set or minimum zero forcing set for the generalized pyramid network.

The following results is a consequence of Theorem 1.9.

**Theorem 2.2.** Let \( G \) be the hypermesh-pyramid network \( P_{HM,n} \), \( n \geq 0 \). Then \( \gamma_{p,k}(G) \geq 2^n - k, \ k \geq 1 \).

**\( k \)-Power Domination in Hypermesh-Pyramid Network**

**Input:** The hypermesh-pyramid network \( P_{HM,n} \), \( n \geq 0 \).

**Algorithm:** Name the vertex in each level \( r, 1 \leq r \leq n \) in the \( i^{th} \) row, \( j^{th} \) column position as \((r, v_{ij})\), \( 1 \leq i \leq 2^r, 1 \leq j \leq 2^r \) and select the vertices \((n,v_{i1}), 1 \leq i \leq 2^n - k \) in \( S \).

**Proof of Correctness:** \( M^0(S) = N[S] \) contains all vertices in the first \( 2^n - k \) rows of \( G \). Each vertex in \( M^0(S) \) is adjacent to \( k \) vertices in \( V(G)\setminus N[S] \). Now vertices in the \( n^{th} \) level are monitored by \( M^1(S) \). Since each vertex in level \( n \) has exactly one neighbour in its proceeding level, vertices in the \( n-1^{th} \) level are monitored by \( M^2(S) \). Preceeding inductively, \( M^{n+1}(S) = V(G) \). Since \( |S| = 2^n - k \) and the subgraph induced by \( S \) is connected, we have \( \gamma_{p,k}^t(G) = \gamma_{p,k}^c(G) = 2^n - k \). □

**Theorem 2.3.** Let \( G \) be the hypermesh-pyramid network \( P_{HM,n} \), \( n \geq 0 \). Then \( \gamma_{p,k}(G) = \gamma_{p,k}^t(G) = \gamma_{p,k}^c(G) = 2^n - k, \ k \geq 1 \).
2.2. Zero Forcing in Hypermesh-Pyramid Network

**Theorem 2.4.** Let \( G \) be the hypermesh-pyramid network \( P_{HM,n} \), \( n \geq 0 \). Then \( Z(G) \geq 4^n \).

**Proof.** Let \( S \) be a zero forcing set of \( P_{HM,n} \). Suppose \( |S| = 4^n - 1 \) and every member of \( S \) is colored blue. Then \( N[S] \) induces a subgraph \( H \) with \( x \) rows and \( y \) columns where \( x, y \leq 4^n - 1 \). Therefore, \( HM_{m,n} \) has \( HM_{\alpha,\beta} \) with \( \alpha \geq 2 \) and \( \beta \geq 2 \). Choose \( u \in N[S] \setminus S \). Then \( u \) is adjacent to at least 2 white vertices in \( HM_{\alpha,\beta} \), a contradiction. Hence \( Z(G) \geq 4^n \). \( \square \)

**Zero Forcing in Hypermesh-Pyramid Network**

**Input:** The hypermesh-pyramid network \( P_{HM,n} \), \( n \geq 0 \).

**Algorithm:** Name the vertex in each level \( r \), \( 1 \leq r \leq n \) in the \( i \)th row, \( j \)th column position as \( (r, v_{ij}) \), \( 1 \leq i \leq 2^r \), \( 1 \leq j \leq 2^r \) and select the vertices \( (n, v_{ij}) \), \( 1 \leq i \leq 2^n \), \( 1 \leq j \leq 2^n \) in \( S \).

**Output:** \( Z(G) = 4^n \).

**Proof of Correctness:** \( S \) contains all vertices in level \( n \) of \( G \). Each vertex in \( S \) is adjacent to exactly one vertex in its preceeding level and hence receives color blue. Therefore \( Z(G) = 4^n \). \( \square \)

**Theorem 2.5.** Let \( G \) be the hypermesh-pyramid network \( P_{HM,n} \), \( n \geq 0 \). Then \( Z(G) = 4^n \).

**Remarks:**
1. Let \( G \) be a grid-like graph with \( \gamma_{p,k}(G) = 1, k \geq 2 \). Then \( \gamma_{p,k}(P_{G,l}) = 1, k \geq 2 \).
2. Let \( G \) be a torus network with \( \gamma_{p,2}(G) = 2 \). Then \( \gamma_{p,2}(P_{G,l}) = 2 \).

**Observation:** Let \( G \) be a graph with \( \gamma_{p,k}(G) = m, k \geq 1, m \geq 1 \). Then \( \gamma_{p,k}(P_{G,l}) = m, k \geq 1, m \geq 1 \).

2.3. Corona Graph

**Definition 2.6.** [1] The corona of two graphs \( G \) and \( H \), denoted by \( G \circ H \) is a graph formed from one copy of \( G \) and \( |V(G)| \) copies of \( H \) where \( i \)th vertex of \( G \) is adjacent to every vertex in the \( i \)th copy of \( H \), \( 1 \leq i \leq |V(G)| \).

**Theorem 2.7.** Let \( G \) be a connected graph of order \( n \geq 2 \) and \( H \) be a graph of order \( m \). Then \( \gamma(G \circ H) = \gamma_t(G \circ H) = \gamma_c(G \circ H) = \gamma_{p,k}(G \circ H) = \gamma_{p,k}^t(G \circ H) = \gamma_{p,k}^c(G \circ H) = n, \forall k \geq 0 \), if \( m \geq k + 1 \).
**Proof.** Let $S$ be a dominating set of $G \circ H$. By definition, $G \circ H$ contains $n$ number of distinct copies of $H$, each of order $m$, say, $H_1, H_2, \ldots, H_n$. In order to dominate, $\cup H_i, 1 \leq i \leq n$, we need at least $n$ vertices. Thus $|S| \geq n$. Now let $S = \{v_1, v_2, \ldots, v_n\} = V(G)$. Then $S$ is a dominating set of $G \circ H$. Therefore, $\gamma(G \circ H) = n$ when $m \geq k + 1$. Now let $S$ be a $k$-power dominating set of $G \circ H$. Suppose $|S| < n$. By definition, there are $n$ vertex disjoint copies of $H_i, 1 \leq i \leq n$ adjacent to each vertex of $G$ in $G \circ H$. Deleting one vertex from $G$ leaves at least one copy of $H$ not monitored by $S$, a contradiction. Thus $|S| \geq n$. Now let $S = \{v_1, v_2, \ldots, v_n\} = V(G)$. Therefore, $\gamma_{p,k}(G \circ H) = n$. Since the subgraph induced by $S$ is connected, we have $\gamma(G \circ H) = \gamma_t(G \circ H) = \gamma_c(G \circ H) = \gamma_{p,k}^t(G \circ H) = \gamma_{p,k}^c(G \circ H) = n$. 


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**References**


