FUZZY DOT $\beta$–IDEALS OF $\beta$–ALGEBRAS

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Abstract: In this paper, we introduce the notion of fuzzy dot $\beta$–ideals on $\beta$–algebras and investigate some of their properties.

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1. Introduction

In 1966, Y. Imai and K. Iseki ([4], [5]) introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. In 2002, J.Neggers and H.S. Kim introduced the notion of $\beta$–algebra [7]. In [6], the author introduced the notion of fuzzy dot subalgebras of $d$-algebras as a generalization of a fuzzy subalgebra. In [1], we introduce the notion of fuzzy $\beta$ subalgebras. In [2], we introduced the notion of fuzzy dot $\beta$–subalgebras of $\beta$–algebras and in [3], we introduced and discussed some properties of fuzzy $\beta$–ideals of $\beta$–algebras. In this paper, we introduce the notion of fuzzy dot $\beta$–ideals of $\beta$–algebras and investigate some of their properties.
2. Preliminaries

In this section we recall some basic definitions that are required in the sequel.

**Definition 2.1.** [7] A \( \beta \)-algebra is a non-empty set \( X \) with a constant 0 and two binary operations + and − satisfying the following axioms:

1. \( x - 0 = x \).
2. \( (0 - x) + x = 0 \).
3. \( (x - y) - z = x - (z + y) \) \( \forall \ x, y, z \in X \).

**Example 2.2.** Let \( X = \{0, 1, 2, 3\} \) be a set with constant 0 and two binary operations + and - are defined on \( X \) with the Cayley’s table

<table>
<thead>
<tr>
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<table>
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Then \( (X, +, −, 0) \) is a \( \beta \)-algebra.

**Note:** In a \( \beta \)-algebra a partial ordering \( \leq \) can be defined by \( x \leq y \) if and only if \( x - y = 0 \).

**Definition 2.3.** Let \( \mu_1 \) and \( \mu_2 \) be two fuzzy sets of \( X_1 \) and \( X_2 \) respectively. Then the direct product \( \mu_1 \times \mu_2 \) of \( \mu_1 \) and \( \mu_2 \) is defined as the fuzzy set of \( X_1 \times X_2 \)

\[
(\mu_1 \times \mu_2)(x_1, x_2) = \min \{\mu_1(x_1), \mu_2(x_2)\} \ \forall \ (x_1, x_2) \in X_1 \times X_2.
\]

**Definition 2.4.** Let \( \mu \) be a fuzzy set in a set \( X \). For \( t \in [0, 1] \), the set \( \mu_t = \{x \in X/\mu(x) \geq t\} \) is called a level subset of \( \mu \).

**Definition 2.5.** A non-empty subset \( A \) of a \( \beta \)-algebra \( (X, +, −, 0) \) is called a \( \beta \)-subalgebra of \( X \), if

1. \( x + y \in A, \forall x, y \in A \) and
2. \( x - y \in A, \forall x, y \in A \).

**Definition 2.6.** ([2], Definition 3.1) Let \( \mu \) be a fuzzy set in a \( \beta \)-algebra \( X \). Then \( \mu \) is called a fuzzy dot \( \beta \)-subalgebra of \( X \) if

1. \( \mu(x + y) \geq \mu(x) \cdot \mu(y) \ \forall x, y \in X \).
2. $\mu(x - y) \geq \mu(x).\mu(y)$ $\forall x, y \in X$.

**Definition 2.7.** ([3], Definition 3.1) A non-empty subset $I$ of a $\beta -$algebra $(X, +, -, 0)$ is called a $\beta -$ideal of $X$, if

1. $0 \in I$,
2. if $x - y$ and $y \in I$ then $x \in I$ $\forall x, y \in X$.

**Definition 2.8.** Let $\mu$ be a fuzzy set in a $\beta -$algebra $X$. Then $\mu$ is called a fuzzy $\beta -$ideal of $X$ if

1. $\mu(0) \geq \mu(x)$ $\forall x \in X$, and
2. $\mu(x) \geq \min\{\mu(x - y), \mu(y)\}$ $\forall x, y \in X$.

### 3. Fuzzy Dot $\beta -$Ideals of $\beta -$Algebras

In this section we introduce the notion of fuzzy dot $\beta -$ideals of $\beta -$algebras and proved some simple theorem.

**Definition 3.1.** Let $\mu$ be a fuzzy set in a $\beta -$algebra $X$. Then $\mu$ is called a fuzzy dot $\beta -$ideal of $X$ if

1. $\mu(0) \geq \mu(x)$ $\forall x \in X$, and
2. $\mu(x) \geq \mu(x - y).\mu(y)$ $\forall x, y \in X$.

**Example 3.2.** Consider the $\beta -$algebra $(X, +, -, 0)$ in Example:2.2

Define $\mu : X \to [0, 1]$ such that

$$
\mu(x) = \begin{cases} 
0.9 & \text{if } x = 0 \\
0.4 & \text{if } x = 1 \\
0.6 & \text{if } x = 2, 3
\end{cases}
$$

then $\mu$ is a fuzzy dot $\beta -$ideal of $X$.

**Theorem 3.3.** Every fuzzy $\beta -$ideal of $X$ is a fuzzy dot $\beta -$ideal of $X$. The converse not true in all cases.

**Proof.** Let $\mu$ is a fuzzy $\beta -$ideal of $X$. Then $\mu(0) \geq \mu(x)$ $\forall x \in X$, and $\mu(x) \geq \min\{\mu(x - y), \mu(y)\} \geq \mu(x - y).\mu(y)$ $\forall x, y \in X$.

Therefore $\mu$ is a fuzzy dot $\beta -$ideal of $X$. 
Note: In the example 3.2, the fuzzy set \( \mu \) is a fuzzy dot \( \beta \)-ideal of \( X \) but \( \mu \) is not a fuzzy \( \beta \)-ideal of \( X \), since \( \mu(1) = 0.4 \not\geq 0.6 = \min \{0.6, 0.6\} = \min \{\mu(3), \mu(2)\} = \min\{\mu(1-2), \mu(2)\}\).

One can easily prove the following.

**Lemma 3.4.** Let \( X \) be a \( \beta \)-algebra, then \( x = (x - y) + y \ \forall \ x \in X \).

**Lemma 3.5.** If \( \mu \) is a fuzzy set of \( X \) such that

1. \( \mu(0) \geq \mu(x^*) \geq \mu(x) \ \forall \ x \in X \) and
2. \( \mu(x + y) \geq \mu(x) \cdot \mu(y) \ \forall \ x, y \in X \),

then \( \mu \) is a fuzzy dot \( \beta \)-ideal of \( X \). Also \( \mu \) is a fuzzy dot \( \beta \)-subalgebra of \( X \).

**Proof.** Using Lemma 3.4 and the given conditions, the result follows.

**Theorem 3.6.** Let \( \mu \) is a fuzzy dot \( \beta \)-ideal of \( X \). Then the following holds.

1. If \( x \leq y \), then \( \mu(x) \geq \mu(0) \cdot \mu(y) \ \forall \ x, y \in X \).
2. If \( x \leq y + z \), then \( \mu(x) \geq \mu(0) \cdot \mu(y) \cdot \mu(z) \ \forall \ x, y, z \in X \).
3. For any positive integer \( n \), \( \mu^n \) is also a fuzzy dot \( \beta \)-ideal of \( X \) where \( \mu^n(x) = (\mu(x))^n \ \forall \ x \in X \).
4. If \( \mu^c \) is a fuzzy dot \( \beta \)-ideal of \( X \), then \( \mu \) is a constant function.

**Proof.**

1. Let \( x, y \in X \). Now

\[
\mu(x) \geq \mu(x - y) \cdot \mu(y) (\because \mu \text{ is a fuzzy dot ideal}) \\
\Rightarrow \mu(x) \geq \mu(0) \cdot \mu(y) (\because x \leq y).
\]

2. follows from definition.

3. Let \( x, y \in X \) and let \( \mu \) is a fuzzy dot \( \beta \)-ideal of \( X \). Then

\[
\mu(0) \geq \mu(x) \Rightarrow (\mu(0))^n \geq (\mu(x))^n \Rightarrow \mu^n(0) \geq \mu^n(x) \\
\mu(x) \geq \mu(x - y) \cdot \mu(y) \Rightarrow (\mu(x))^n \geq (\mu(x - y))^n \cdot (\mu(y))^n \\
\text{Thus, } \mu^n(x) \geq \mu^n(x - y) \cdot \mu^n(y)
\]
4. For any $x \in X$,
   $\mu(0) \geq \mu(x)$ (since $\mu$ is a fuzzy dot $\beta$–ideal)

   Now $\mu^{c}$ is a fuzzy dot $\beta$–ideal.

   Thus $\mu^{c}(0) \geq \mu^{c}(x) \Rightarrow 1 - \mu(0) \geq 1 - \mu(x) \Rightarrow \mu(0) \leq \mu(x)$

   Hence, $\mu(0) = \mu(x) \ \forall \ x \in X$, $\mu$ is a constant function.

**Theorem 3.7.** If $\mu_{1}$ and $\mu_{2}$ be two fuzzy dot $\beta$–ideals of $X$ then $\mu_{1} \cap \mu_{2}$

is also a fuzzy dot $\beta$–ideal of $X$.

The above theorem can be generalized as follows.

**Corollary 3.8.** If $\{\mu_{i} / i = 1, 2, 3, \cdots \}$ be a family of fuzzy dot $\beta$–ideals of $X$,
then $\cap \mu_{i}$ is also a fuzzy dot $\beta$–ideal of $X$.

**Theorem 3.9.** Let $\mu_{1}$ and $\mu_{2}$ be two fuzzy dot $\beta$–ideals of $\beta$–algebra $X$.

Then the direct product $\mu_{1} \times \mu_{2}$ of $\mu_{1}$ and $\mu_{2}$ is defined by $(\mu_{1} \times \mu_{2})(x, y) =
\mu_{1}(x) \cdot \mu_{2}(y)$ is also a fuzzy dot $\beta$–ideal of $X \times X$.

**Proof.** Let $X = X \times X$ and $\mu = \mu_{1} \times \mu_{2}$.

Let $x = (x_{1}, x_{2})$ and $y = (y_{1}, y_{2})$ be two elements of $X$.

Clearly $\mu(\theta) \geq \mu(x) \forall x = (x_{1}, x_{2})$ and $\theta = (0, 0) \in X$.

$\mu(x) = \mu((x_{1}, x_{2}) = \mu_{1}(x_{1}) \cdot \mu_{2}(x_{2})$ Thus

$$
\mu(x) \geq \mu_{1}(x_{1} - y_{1}) \cdot \mu_{2}(x_{2} - y_{2}) \cdot \mu_{2}(y_{2})
$$

$$
= (\mu_{1} \times \mu_{2})(x_{1}, x_{2}) - (y_{1}, y_{2})) \cdot (\mu_{1} \times \mu_{2})(y_{1}, y_{2})
$$

(Since $\mu_{1} \mu_{2}$ are fuzzy dot ideals)

$$
= \mu(x - y) \cdot \mu(y).
$$

Hence $\mu_{1} \times \mu_{2}$ is a fuzzy dot $\beta$–subalgebra of $X \times X$.

The above result can be extended as follows.

**Theorem 3.10.** Let $\mu_{1}$ and $\mu_{2}$ be two fuzzy dot $\beta$–ideals of $\beta$–algebra $X_{1}$ and $X_{2}$ respectively.

Then the direct product $\mu_{1} \times \mu_{2}$ is a fuzzy dot $\beta$–ideal of $X_{1} \times X_{2}$.

**Theorem 3.11.** Let $f : X \rightarrow Y$ be a homomorphism of a $\beta$–algebra $X$

into a $\beta$–algebra $Y$. If $\mu$ is a fuzzy dot $\beta$–ideal of $Y$, then the pre-image of $\mu$

, denoted by $f^{-1}(\mu)$ is defined as $\{f^{-1}(\mu)\}(x) = \mu(f(x)), \forall x \in X$, is a fuzzy dot $\beta$–ideal of $X$.

**Proof.** Let $\mu$ be a fuzzy dot $\beta$–ideal of $Y$. For all

$x \in X, f^{-1}(\mu)(0) = \mu(f(0)) \geq \mu(f(x)) = \{f^{-1}(\mu)\}(x)$ Let $x, y \in X$. Then

$$
\{f^{-1}(\mu)\}(x) = \mu(f(x)) \geq \mu(f(x - y)) \cdot \mu(f(y)) = \{f^{-1}(\mu)(x - y)\} \cdot \{f^{-1}(\mu)\}(y)
$$
Hence $f^{-1}(\mu)$ is a fuzzy dot $\beta$–ideal of $X$.

**Theorem 3.12.** Let $f : X \to Y$ be an epimorphism of a $\beta$–algebra $X$ into a $\beta$–algebra $Y$. Then $\mu$ is a fuzzy dot $\beta$–ideal of $Y$, whenever $f^{-1}(\mu)$ is a fuzzy dot $\beta$–ideal of $X$.

**Proof.** For any $y \in Y$, there exist a $x \in X$ such that $f(x) = y$. Then $\mu(y) = \mu(f(x)) = \{f^{-1}(\mu)\}(x) \leq \{f^{-1}(\mu)\}(0) = \mu(f(0)) = \mu(0)$. Let $y, y' \in Y$.

Then there exist some $x, x' \in X$ such that $f(x) = y$ and $f(x') = y'$. Now

$$
\mu(y) = \mu(f(x)) \\
= \{f^{-1}(\mu)\}(x) \\
\geq \{f^{-1}(\mu)\}(x-x') \cdot \{f^{-1}(\mu)\}(x') \\
= \mu(f(x-x') \cdot \mu(f(x'))) \\
= \mu(f(x) - f(x')) \cdot \mu(f(x')) \\
= \mu(y - y') \cdot \mu(y').
$$

Hence $\mu$ is a fuzzy dot $\beta$–ideal of $Y$.

**References**


