ORTHOGONAL \((\alpha, \beta)\) DERIVATIONS ON SEMIRINGS

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Abstract: In this paper, we introduce the notion of orthogonal \((\alpha, \beta)\) derivation on semirings and prove some results on semiprime semirings.

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1. Introduction

The notion of semirings was first introduced in 1934 by H.S.Vandiver[11], while studying the algebra of ideals in rings. In [3],[4] Chandramouleeswaran and Thiruveni studied the notion of derivations on semirings.

In [2], Bresar and Vukuman introduced the notion of orthogonality for a pair \(d, g\) of derivations on a semiprime ring and they gave several necessary and sufficient conditions for \(d\) and \(g\) to be orthogonal. In [1], Aydin N. and Kaya K.discussed the notion of \((\alpha, \beta)\) derivation on prime rings. In [10] the author introduced the notion of \((\alpha, \beta)\) derivation on semirings. In [8],[9] we introduced the notion of orthogonal derivations and orthogonal generalized derivation on semirings. Motivated by these works, in this paper, we define and discuss the notion of orthogonal \((\alpha, \beta)\) derivation on semirings.

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2. Preliminaries

In this section, we recall some basic definitions and results that are needed for our work.

**Definition 2.1.** A semiring is a nonempty set $S$ on which operations of addition and multiplication have been defined such that

1. $(S, +)$ is a commutative semigroup
2. $(S, \cdot)$ is a semigroup
3. Multiplication distributes over addition from either side

**Definition 2.2.** Let $(S, +, \cdot)$ be a semiring. An element $a$ in $S$ is called additively left cancellative if $a + b = a + c \implies b = c \forall b, c \in S$. It is said to be right cancellative if $b + a = c + a \implies b = c$. The element $a \in S$ is said to be additively cancellative if it is both left and right cancellative. The Semiring $S$ is said to be additively cancellative if all the elements in $S$ are additively cancellative

**Definition 2.3.** A semiring $(S, +, \cdot)$ is said to be a semiring with zero, if it has an element 0 in $S$ such that $x + 0 = 0 = 0 + x$ and $x \cdot 0 = 0 = 0 \cdot x \forall x \in S$.

**Definition 2.4.** Let $S$ be a semiring.

1. $S$ is said to be prime if $aSb = 0 \implies a = 0$ or $b = 0$.
2. $S$ is said to be semiprime if $aSa = 0 \implies a = 0$.
3. $S$ is said to be 2 torsion free if $2a = 0, a \in S \implies a = 0$.

**Definition 2.5.** An additive mapping $d : S \to S$ is called a derivation if
\[
d(xy) = d(x)y + xd(y) \forall x, y \in S.
\]

**Definition 2.6.** Let $\alpha$ and $\beta$ be two automorphisms of a semiring $S$. An additive mapping $d : S \to S$ is called a $(\alpha, \beta)$ derivation if
\[
d(xy) = \alpha(x)d(y) + d(x)\beta(y), \forall x, y \in S.
\]

**Lemma 2.7.** [8] Let $S$ be a 2-torsion free semiprime semiring, $a$ and $b$ the elements of $S$. Then the following are equivalent.

1. $aSb = 0$
2. \( bSa = 0 \)

3. \( aSb + bSa = 0 \)
   If one of these conditions are fulfilled then \( ab = ba = 0 \)

**Lemma 2.8.** [8] Let \( S \) be a 2-torsion free semiprime semiring. Suppose that additive mappings \( f \) and \( g \) of a semiring \( S \) into \( S \) satisfies \( f(x)Sg(x) = 0 \), \( \forall x \in S \) then \( f(x)Sg(y) = 0 \) \( \forall x, y \in S \).

**Notation:** Throughout this paper we assume \( S \) to be the semiring with 0 and 1 and additively cancellative.

### 3. Orthogonal \((\alpha, \beta)\) Derivations on Semirings

In this section we introduce the notion of orthogonal \((\alpha, \beta)\) derivations on semirings.

**Definition 3.1.** Let \( S \) be a semiring. The \((\alpha, \beta)\) derivations \( d \) and \( g \) on \( S \) is said to be orthogonal if \( d(x)Sg(y) = g(x)Sd(y) \) \( \forall x, y \in S \)

**Lemma 3.2.** Let \( S \) be a semiprime semiring and \( d, g \) be \((\alpha, \beta)\) derivations on \( S \). Then \( d \) and \( g \) are orthogonal iff \( d(x)g(y) + g(x)d(y) = 0 \).

**Proof.** Assume
\[
d(x)g(y) + g(x)d(y) = 0. \tag{3.1}
\]
Replacing \( y \) by \( yx \) we get \( d(x)\alpha(y)g(x) + g(x)\alpha(y)d(x) \)
Since \( \alpha \) is an automorphism we have \( d(x)Sg(x) + g(x)Sd(x) = 0 \) \( \forall x \in S \).
Using Lemma 2.7 and 2.8 we arrive that \( d \) and \( g \) are orthogonal.
Conversely let \( d \) and \( g \) are orthogonal
Again By lemma 2.7, we have \( d(x)g(y) + g(x)d(y) = 0 \) \( \forall x, y \in S \).

**Lemma 3.3.** Let \( S \) be a semiprime semiring and \( d, g \) be \((\alpha, \beta)\) derivations on \( S \) such that \( g(x)\beta(x) = \beta(x)g(x) \). Then \( d \) and \( g \) are orthogonal iff \( d(x)g(x) = 0 \).

**Proof.** Suppose
\[
d(x)g(x) = 0 \tag{3.2}
\]
Replacing \( x \) by \( x + y \) and simplifying,we get \( d(x)g(y) + d(y)g(x) \)
Again replacing \( y \) by \( yx \) we get \( d(x)\alpha(y)g(x) \)
Since \( \alpha \) is an automorphism it implies \( d(x)Sg(x) = 0 \).
Then by Lemma 2.7 and Lemma 2.8, we get the fact that $d$ and $g$ are orthogonal.

Conversely let $d$ and $g$ are orthogonal.
By lemma 2.7 and replacing $y$ by $x$ we get $d(x)g(x) = 0 \forall x \in S$.

**Theorem 3.4.** Let $S$ be a 2 torsionfree semiprime semiring and $d, g$ be $(\alpha, \beta)$ derivations on $S$ such that $d\alpha = \alpha d, d\beta = \beta d, g\alpha = \alpha g, g\beta = \beta g$ Then the following are equivalent.

(i) $d$ and $g$ are orthogonal
(ii) $dg = 0$
(iii) $gd = 0$
(iv) $dg + gd = 0$
(v) $dg$ is a $(\alpha^2, \beta^2)$ derivation on $S$.

(i) $\iff$ (ii)
Suppose $d$ and $g$ are orthogonal. Then $0 = d(\alpha(x))\alpha(s)dg(y)$
Since $\alpha$ is an automorphism, we have $d(x_1)Sdg(y) = 0 \forall x_1, y \in S$.
Replacing $x_1$ by $g(y)$ we have $dg(y)Sdg(y) = 0$.

Since $S$ is semiprime, this implies that $dg(y) = 0 \forall y \in S \Rightarrow dg = 0$

Conversely, let $dg = 0$. Then $0 = d(\alpha(x))g(\beta(y)) + g(\alpha(x))d(\beta(y))$
Since $\alpha, \beta$ are automorphisms we have $d(x_1)g(y_1) + g(x_1)d(y_1) = 0$ By lemma 3.2, $d$ and $g$ are orthogonal.

Analogously we can prove (i) $\iff$ (iii)

(i) $\iff$ (iv)
Suppose $dg + gd = 0$. Then $2(d(\alpha(x))g(\beta(y)) + g(\alpha(x))d(\beta(y))$
Since $S$ is 2 - torsion free, $\alpha$ and $\beta$ are automorphisms we have $d(x_1)g(y_1) + g(x_1)d(y_1) = 0$

By lemma 3.2, we get $d$ and $g$ are orthogonal.

Conversely suppose that $d$ and $g$ are orthogonal.

By (ii) and (iii), we have $dg = 0 = gd$ That is $dg + gd = 0$.

(v) $\iff$ (i)
Suppose $dg$ is a $(\alpha^2, \beta^2)$ derivation on $S$. That is

$$dg(xy) = \alpha^2(x)dg(y) + d(x)\beta^2(y) \quad(3.3)$$

But $dg(xy) = d(\alpha(x)g(y) + g(x)\beta(y))$. Hence

$$dg(x) = \alpha^2(x)dg(y) + d\alpha(x)\beta g(y) + \alpha g(x)\beta d(y)d(x)\beta^2(y) \quad(3.4)$$
Comparing 3.3, 3.4 we have
\[ d\alpha(x)\beta g(y) + \alpha g(x)d(\beta(y)) = 0 \]
That is
\[ d(\alpha(x)g(\beta(y))) + g(\alpha(x))d(\beta(y)) = 0 \]
Since \( \alpha \) and \( \beta \) are automorphisms we have
\[ d(x_1)g(y_1) + g(x_1)d(y_1) = 0 \quad \forall \ x, y \in S. \]
By lemma 3.2, \( d \) and \( g \) are orthogonal.
Conversely suppose that \( d \) and \( g \) are orthogonal. Then by (ii), we have
\[ dg = 0 \]
Hence \( dg \) is a \((\alpha^2, \beta^2)\) derivation on \( S \).

References