Approximation of Signals (Functions) by
\((E,q)(C,1)\) Product Operators

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Abstract: In this paper, we establish a quite new theorem on degree of approximation of a function \(\tilde{f}\), conjugate to a \(2\pi\) periodic function \(f\) belonging to class \(\text{Lip}_\alpha\) by \((E,q)(C,1)\) product operators on a conjugate Fourier series.

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1. Introduction

Let \(\sum_{n=0}^{\infty} u_n\) be a given infinite series with \(s_n\) for its \(n^{th}\) partial sum.
Let \( \{t^E_n\} \) denote the sequence of \((E,q_n)\) mean of the sequence \(\{s_n\}\). If the \((E,q_n)\) transform of \(s_n\) is defined as
\[
t^E_n(f;x) = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k(f;x) \rightarrow s \quad \text{as} \ n \rightarrow \infty
\] (1)
the series \( \sum_{n=0}^{\infty} u_n \) is said to be summable to the number \(s\) by the \((E,q_n)\) method (Hardy [8]).

The \((E,q_n)\) transform reduces to the \((E,1)\) transform if for all \(q_n = 1\).

Let \(\{t^C_n\}\) denote the sequence of \((C,1)\) mean of the sequence \(\{s_n\}\). If the \((C,1)\) transform of \(s_n\) is defined as
\[
t^C_n(f;x) = \frac{1}{n+1} \sum_{k=0}^{n} s_k(f;x) \rightarrow s \quad \text{as} \ n \rightarrow \infty
\] (2)
the series \( \sum_{n=0}^{\infty} u_n \) is said to be summable to the number \(s\) by \((C,1)\) method (Cesàro method).

Thus if
\[
t^{EC}_n(f;x) = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{1}{k+1} \sum_{\nu=0}^{k} s_{\nu}(f;x) \rightarrow s \quad \text{as} \ n \rightarrow \infty,
\] (3)
where \(\{t^{EC}_n\}\) denote the sequence of \((E,q_n)(C,1)\) product mean of the sequence \(s_n\), the series \( \sum_{n=0}^{\infty} u_n \) is said to be summable to the number \(s\) by \((E,q_n)(C,1)\) method.

Let \(f\) be a \(2\pi\)-periodic function and Lebesgue integrable. The Fourier series associated with \(f\) at a point \(x\) is defined by
\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x)
\] (4)
with partial sums \(s_n(f;x)\).

The conjugate series of Fourier series (4) of \(f\) is given by
\[
\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \equiv \sum_{n=1}^{\infty} B_n(x)
\] (5)
with partial sums \(\tilde{s}_n(f;x)\).

Throughout this paper, we will call (5) as conjugate Fourier series of function \(f\).
\[ L_\infty - \text{ norm of a function } f : \mathbb{R} \rightarrow \mathbb{R} \text{ is defined by} \]
\[ \| f \|_\infty = \sup \{ |f(x)| : x \in \mathbb{R} \} \quad (6) \]

\[ L_r- \text{ norm is defined by} \]
\[ \| f \|_r = \left( \int_0^{2\pi} |f(x)|^r \, dx \right)^{\frac{1}{r}} \text{ for some } r \geq 1. \quad (7) \]

The degree of approximation of a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) by a trigonometric polynomial \( T_n(x) \) of degree \( n \) under sup norm \( \| \cdot \|_\infty \) is defined by
\[ \| f(x) - T_n(x) \|_\infty = \sup \{ |f(x) - T_n(x)| : x \in \mathbb{R} \} \text{ (Zygmund [2])} \quad (8) \]

and \( E_n(f) \) of a function \( f \in L_r \) is given by
\[ E_n(f) = \min_{T_n} \| f(x) - T_n(x) \|_r \quad (9) \]

This method of approximation is called Trigonometric Fourier Approximation (TFA).

A function \( f \in \text{Lip}_\alpha \) if
\[ | f(x + t) - f(x) | = O(|t|^{\alpha}) \text{ for } 0 < \alpha \leq 1. \quad (10) \]

We shall use the following notations:
\[ \psi(t) = \psi(x,t) = f(x + t) - f(x - t) \]
\[ \tilde{K}_n(t) = \frac{1}{2\pi (1 + q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{1}{1 + k} \sum_{\nu=0}^{k} \frac{\cos(\nu + \frac{1}{2}) t}{\sin \frac{\nu}{2}} \]

2. The Main Result

Several researchers ([3], [4], [5], [6], [7], [9], [10], [11], [12]) studied error estimates \( E_n(f) \) using different linear operators. In the present paper, we establish a theorem on the degree of approximation of a function \( \tilde{f} \) conjugate to a periodic function \( f \) belonging to the class \( \text{Lip}_\alpha \) by \( (E,q_n)(C,1) \) product mean on its conjugate Fourier series in the following form:
Theorem 1. If \( \{t_n^{EC}\} \) denote the sequence of \((E,q_n)(C,1)\) product mean of the sequence \(\{s_n\}\) and the conjugate Fourier series given by \(\hat{f}\) converges at the point \(x\) to the value which is denoted by

\[
\hat{f}(x) = -\frac{1}{2\pi} \int_0^{2\pi} \psi(t) \cot\left(\frac{t}{2}\right) dt \quad (Zygmund [1])
\] (11)

then the degree of approximation of a function \(\hat{f}\) conjugate to a \(2\pi\)-periodic function \(f\) belonging to the class \(\text{Lip}_\alpha\) by \((E,q_n)(C,1)\) mean on its conjugate Fourier series (5) is given by

\[
\left\| \hat{f}(x) - t_n^{EC}(x) \right\|_\infty = O\left\{ \frac{1}{(n+1)^\alpha} \right\} \quad \text{for } 0 < \alpha < 1.
\] (12)

3. Lemmas

For the proof of our theorems, following lemmas are required:

Lemma 1. For \(0 \leq t \leq \frac{1}{n+1}\),

\[
\left| K_n(t) \right| = O\left( \frac{1}{t} \right)
\]

Proof. For \(0 \leq t \leq \frac{1}{n+1}\), \(\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}\) and \(|\cos nt| \leq 1\)

\[
\left| K_n(t) \right| = \frac{1}{2\pi (1+q)^n} \sum_{k=0}^{n} \left\{ \binom{n}{k} q^{n-k} \frac{1}{(1+k)} \sum_{\nu=0}^{k} \frac{\cos \left(\nu + \frac{1}{2}\right) t}{\sin \left(\frac{t}{2}\right)} \right\} \]

\[
\leq \frac{1}{2t (1+q)^n} \sum_{k=0}^{n} \left\{ \binom{n}{k} q^{n-k} \left( \frac{1}{1+k} \right) \sum_{\nu=0}^{k} 1 \right\} \]

\[
\leq \frac{1}{2t (1+q)^n} \sum_{k=0}^{n} \left\{ \binom{n}{k} q^{n-k} \right\} \]

\[
= \frac{1}{2t (1+q)^n} (1+q)^n
\]
\[ = O \left( \frac{1}{t} \right) \] since \( \sum_{k=0}^{n} \binom{n}{k} q^{n-k} = (1 + q)^n. \]

**Lemma 2.** For \( 0 \leq a \leq b \leq \infty, 0 \leq t \leq \pi \) and any \( n \), we have

\[ |\tilde{K}_n(t)| = O \left( \frac{1}{t} \right). \]

**Proof.** For \( 0 \leq \frac{1}{n+1} \leq t \leq \pi \), by applying Jordan’s lemma \( \sin \left( \frac{\pi}{2} \right) \geq \frac{t}{\pi} \).

\[
|\tilde{K}_n(t)| = \frac{1}{2\pi (1+q)^n} \left| \sum_{k=0}^{n} \left( \binom{n}{k} q^{n-k} \frac{1}{1+k} \sum_{\nu=0}^{k} \frac{\cos \left( \nu + \frac{1}{2} \right) t}{\sin \frac{\pi}{2}} \right) \right|
\]
\[
\leq \frac{1}{2t (1+q)^n} \left| \sum_{k=0}^{n} \left( \binom{n}{k} q^{n-k} \frac{1}{1+k} \Re \left\{ \sum_{\nu=0}^{k} e^{i \nu t} \right\} \right) \right| e^{i\frac{\pi}{2}}
\]
\[
\leq \frac{1}{2t (1+q)^n} \left| \sum_{k=0}^{n} \left( \binom{n}{k} q^{n-k} \frac{1}{1+k} \Re \left\{ \sum_{\nu=0}^{k} e^{i \nu t} \right\} \right) \right|
\]
\[
\leq \frac{1}{2t (1+q)^n} \left| \sum_{k=0}^{n} \left( \binom{n}{k} q^{n-k} \frac{1}{1+k} \Re \left\{ \sum_{\nu=0}^{k} e^{i \nu t} \right\} \right) \right|
\]
\[
+ \frac{1}{2t (1+q)^n} \left| \sum_{k=\tau}^{n} \left( \binom{n}{k} q^{n-k} \frac{1}{1+k} \Re \left\{ \sum_{\nu=0}^{k} e^{i \nu t} \right\} \right) \right|
\]

where \( \tau \) denotes the integral part of \( \frac{1}{t} \).

Now considering first term of (13),

\[
\frac{1}{2t (1+q)^n} \left| \sum_{k=0}^{\tau-1} \left( \binom{n}{k} q^{n-k} \frac{1}{1+k} \Re \left\{ \sum_{\nu=0}^{k} e^{i \nu t} \right\} \right) \right|
\]
\[
\leq \frac{1}{2t (1+q)^n} \left| \sum_{k=0}^{\tau-1} \left( \binom{n}{k} q^{n-k} \frac{1}{1+k} \left\{ \sum_{\nu=0}^{k} 1 \right\} \right) \right| e^{i\nu t}
\]
\[
\leq \frac{1}{2t (1+q)^n} \left| \sum_{k=0}^{\tau-1} \left( \binom{n}{k} q^{n-k} \right) \right|
\]

Now considering second term of (13) and using Abel’s lemma
\[
\frac{1}{2t (1 + q)^n} \left| \sum_{k=\tau}^{n} \left( \binom{n}{k} q^{n-k} \frac{1}{1+k} \left. Re \left\{ \sum_{\nu=0}^{k} e^{i\nu t} \right\} \right. \right| \right.
\leq \frac{1}{2t (1 + q)^n} \sum_{k=\tau}^{n} \left( \binom{n}{k} q^{n-k} \frac{1}{1+k} \max_{0 \leq m \leq k} \left| \sum_{\nu=0}^{k} e^{i\nu t} \right| \right.
\leq \frac{1}{2t (1 + q)^n} \sum_{k=\tau}^{n} \left( \binom{n}{k} q^{n-k} \frac{1}{1+k} \left( 1 + k \right) \right.
= \frac{1}{2t (1 + q)^n} \sum_{k=\tau}^{n} \left( \binom{n}{k} q^{n-k} \right.
\]  

(15)

Combining (13), (14) and (15), we get

\[
\left| \tilde{K}_n (t) \right| \leq \frac{1}{2t (1 + q)^n} \sum_{k=\tau}^{\tau-1} \left( \binom{n}{k} q^{n-k} \right) + \frac{1}{2t (1 + q)^n} \sum_{k=\tau}^{n} \left( \binom{n}{k} q^{n-k} \right)
= O \left( \frac{1}{t} \right).
\]

\[
4. \text{ Proof of the Theorem}
\]

It is well known that

\[
\tilde{s}_n (f; x) - \tilde{f} (x) = \frac{1}{2\pi} \int_{0}^{\pi} \psi (t) \frac{\cos \left( n + \frac{1}{2} \right) t}{\sin \left( \frac{t}{2} \right)} \, dt
\]

Using (5) the \((C, 1)\) transform of \(\tilde{s}_n (f; x)\) is given by

\[
\tilde{f} (x) - t_n^{C} (x) = \frac{1}{2\pi (n + 1)} \int_{0}^{\pi} \psi (t) \sum_{k=0}^{n} \frac{\cos \left( k + \frac{1}{2} \right) t}{\sin \left( \frac{t}{2} \right)} \, dt
\]

The \((E, q_n)(C, 1)\) transform of \(\tilde{s}_n (f; x)\) is given by

\[
\tilde{f} (x) - t_n^{EC} (x)
= \frac{1}{2\pi (1 + q)^n} \sum_{k=0}^{n} \left[ \left( \binom{n}{k} q^{n-k} \int_{0}^{\pi} \psi (t) \frac{1}{\sin \frac{t}{2}} \left( \frac{1}{k+1} \right) \left\{ \sum_{\nu=0}^{k} \cos \left( \nu + \frac{1}{2} \right) t \right\} \right. \, dt \right]
= \int_{0}^{\pi} \psi (t) \, \tilde{K}_n (t) \, dt
\]
\[
\hat{f}(x) - t_n^{EC}(x) = I_{1.1} + I_{1.2} \quad \text{(say)}
\]

Using Lemma 1, we have

\[
| I_{1.1} | \leq \int_0^{\frac{1}{n+1}} | \psi(t) | \left| \tilde{K}_n(t) \right| dt
= O \int_0^{\frac{1}{n+1}} \frac{t^\alpha}{| t |} dt
= O \int_0^{\frac{1}{n+1}} t^{\alpha-1} dt
= O \left\{ \frac{1}{\alpha} \right\} \frac{1}{n+1} dt
I_{1.1} = O \left\{ \frac{1}{(n+1)^\alpha} \right\}
\]

Using Lemma 2, we have

\[
| I_{1.2} | \leq \int_{\frac{1}{n+1}}^\pi | \psi(t) | \left| \tilde{K}_n(t) \right| dt
= O \int_{\frac{1}{n+1}}^\pi \frac{t^\alpha}{| t |} dt
= O \int_{\frac{1}{n+1}}^\pi t^{\alpha-1} dt
= O \left\{ \frac{1}{\alpha} \right\} \frac{\pi}{\frac{1}{n+1}} dt
I_{1.2} = O \left\{ \frac{1}{(n+1)^\alpha} \right\}
\]

This completes the proof of the theorem.

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References


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