Boris Khots and Dmitriy Khots

QUANTUM MECHANICS
FROM OBSERVER'S MATHEMATICS POINT OF VIEW

Academic Publications, 2015

Boris Khots and Dmitriy Khots

QUANTUM MECHANICS
FROM OBSERVER'S MATHEMATICS POINT OF VIEW

Address: 4425 Mills Civic Parkway, unit 1202,
West Des Moines, Iowa 50265, USA
bkhots@mchsi.com

DOI: 10.12732/acadpublmon2015007

http://acadpubl.eu/ap/

Academic Publications, 2015

Dedicated to the memory of Solomon and Olga Khots

Our parents and grandparents

Contents

1. FORWARD

This book considers and summarizes authors' results, which were done in last 10 years for so-called Observer's Mathematics. This mathematics were introduced and published by authors in 2004. After that authors presented and published more than 50 research papers with investigation of various properties of this mathematics and its applications to classical mathematics itself and to contemporary physics including quantum mechanics.

When we consider and analyze physical events with the purpose of creating corresponding models we often assume that the mathematical apparatus used in modeling is infallible. In particular, this relates to the use of infinity in various aspects and the use of Newton's definition of a limit in analysis. First of all we define Observer's Mathematics which we have created to avoid infinity idea and consider as an alternative to contemporary mathematics. Observer dependent ascending chain of embedded sets of decimal fractions and their Cartesian products are considered. For every set, arithmetic operations are defined (these operations locally coincide with standard operations). The basic problems of Algebra, Geometry, Topology, and Logic are solved for this chain. Definition of Dimension of these sets is introduced. Euclidean, Lobachevsky, and Riemannian Geometries become the particular cases of the developed Geometry, although many others are possible.

Certain results that have been predicted by Quantum Mechanics (QM) theory are not always supported by experiments. This defines a deep crisis in contemporary physics and, in particular, quantum mechanics. We believe that, in fact, the mathematical apparatus employed within today's physics is a possible reason. This book is an attempt to relay how Observer's Mathematics may explain some of the contradictions in QM theory results. We consider the Hamiltonian Mechanics, Newton equation, Schrodinger equation, two slit interference, wave-particle duality for single photons, uncertainty principle, Dirac equations for free electron, photoelectric effect, and Lorentz transformation in a setting of arithmetic, algebra, and topology provided by Observer's Mathematics.

2. INTRODUCTION TO OBSERVER'S MATHEMATICS

The main definitions and properties of Observer's Mathematics were initially published in 1, 2, and 3.

2.1 Observers

We consider a finite well-ordered system of observers, where each observer sees the real numbers as the set of all infinite decimal fractions. The observers are ordered by their level of "depth", i.e. each observer has a depth number (hence, we have the regular integer ordering), such that an observer with depth k sees that an observer with depth $n < k$ sees and deals (to be defined below) not with an infinite set of infinite decimal fractions, but, actually, with a finite set of finite decimal fractions. We call this set W_n, i.e. it is the set of all decimal fractions, such that there are at most n digits in the integer part and n digits in the decimal part of the fraction. Visually, an element in W_n looks like $\underbrace{\_ \ldots \_}_{n} . \underbrace{\_ \ldots \_}_{n}$. Moreover, an observer with a given depth is unaware (or can only assume the existence) of observers with larger depth values and for his purposes, he deals with "infinity". These observers are called *naive*, with the observer with the lowest depth number – the most naive. However, if there is an observer with a higher depth number, he sees that a given observer actually deals with a finite set of finite decimal fractions, and so on. Therefore, if we fix an observer, then this observer sees the sets W_{n_1} ,. . . , W_{n_k} with $n_1 < \ldots < n_k$ indicating the depth level, and realizes that the corresponding observers see and deal with infinity. When we talk about observers, we shall always have some fixed observer (called 'us') who oversees all others and realizes that they are naive. The "W_n-observer" is the abbreviation for somebody who *deals* with W_n while thinking that he deals with infinity.

2.2 Arithmetic

We begin by defining sets W_n which consist of all finite decimal fractions such that there are at most n digits in the integer part and at most n digits in the decimal part. That is, the set W_n contains all elements of the form $a = a_0.a_1...a_n$ where the integer part can be written as $a_0 = b_{n-1}...b_0$, where $b_{n-1},...,b_0,a_1,....,a_n \in \{0,1,...,9\}$. If $n < m$, then W_n naturally embeds into W_m by placing 0's in the $n+1^{st}$ through m^{th} decimal places. We call the embedding $\varphi_{n,m} : W_n \to W_m$. Here are some examples: let $2.34 \in W_2$ and then $\varphi_{2,4}(2.34) = 2.3400 \in W_4$. We can also write $W_n \subset W_m$ for $n < m$.

Now, given $c = c_0.c_1...c_n$, $d = d_0.d_1...d_n \in W_n$ we endow W_n with the following arithmetic $(+_n, -_n, \times_n, \div_n)$:

DEFINITION 2.1. *Addition and subtraction*

$$c \pm_n d = \begin{cases} c \pm d, \text{ if } c \pm d \in W_n \\ \text{not defined, if } c \pm d \notin W_n \end{cases}$$

where $c \pm d$ *is the standard addition and subtraction, and we write* $((...(f_1 +_n f_2)...) +_n f_N) = \sum_{i=1}^{N}{}^n f_i$ *for* $f_1,...,f_N$ *iff the contents of any parenthesis are in* W_n, $f_1,\dots,f_N \in W_n$.

DEFINITION 2.2. *Multiplication*

$$c \times_n d = \sum_{k=0}^{n}{}^n \sum_{m=0}^{n-k}{}^n 0.\underbrace{0...0}_{k-1}c_k \cdot 0.\underbrace{0...0}_{m-1}d_m$$

where $c,d \geq 0$, $c_0 \cdot d_0 \in W_n$, $0.\underbrace{0...0}_{k-1}c_k \cdot 0.\underbrace{0...0}_{m-1}d_m$ *is the standard product, and* $k = m = 0$ *means that* $0.\underbrace{0...0}_{k-1}c_k = c_0$ *and* $0.\underbrace{0...0}_{m-1}d_m = d_0$. *If either* $c < 0$ *or* $d < 0$, *then we compute* $|c| \times_n |d|$ *and define* $c \times_n d = \pm |c| \times_n |d|$, *where the sign* $\pm$ *is defined as usual. Note, if the content of at least one parentheses (in previous formula) is not in* W_n, *then* $c \times_n d$ *is not defined.*

Definition 2.3. *Division*

$$c \div_n d = \begin{cases} r, \text{ if } \exists\, r \in W_n \quad r \times_n d = c \\ \\ \text{not defined, if no such } r \text{ exists} \end{cases}$$

Let $n = 2$, so we are in W_2. Here are some examples of elements of W_2: $3.14, -99, 0.1 \in W_2$ and $0.115, 123.9, -100000 \notin W_2$. Now, the examples of arithmetic: $2.08 +_2 11.9 = 13.98$; $(-2.08) +_2 11.9 = 9.82$; $80 +_2 24 =$ not defined; $21.36 -_2 0.87 = 20.49$; $1.36 -_2 16.95 = -15.59$; $1.36 -_2 (-99.95) =$ not defined; $11 \times_2 8 = 88$; $(-5) \times_2 19 = -95$; $11 \times_2 12 =$ not defined; $3.41 \times_2 2.64 = 8.98$; $3.41 \times_2 (-2.64) = -8.98$; $3.41 \times_2 42.64 =$ not defined; $99.41 \times_2 1.64 =$ not defined; $0.85 \times_2 0.02 = 0$; $80 \div_2 4 = 20$; $1 \div_n 0.5 = \{2, 2.01, 2.02, 2.04, 2.05, 2.06, 2.07, 2.08, 2.09\}$ - we get 10 different r's; $1 \div_n 3 =$ not defined (since no r exists). In case $p > q$, $\star \to \infty$ for W_q-observer means $\star \to 10^q$ for W_p-observer, and $\star \to 0$ for W_q-observer means $\star \to 10^{-q}$ for W_p-observer.

Note, if $a = a_0.a_1...a_n \in W_n$ and $b = b_0.b_1...b_n \in W_n$, then $a, b \in W_m$ with $m > n$ by above and the operation of addition in W_n and W_m are equivalent, i.e. $a +_n b = a +_m b$ (if $a \times_n b$ is defined); however, multiplication is not equivalent, $a \times_n b \neq a \times_m b$. For example, if $a = 2.14$, $b = 0.17 \in W_2 \subset W_4$, then $a +_2 b = a +_4 b = 2.31$ and $a \times_2 b = 0.35 \neq a \times_4 b = 0.3638$.

Now, let us look at some main properties of W_n. The following conditions are satisfied:

1. $\forall x \in W_k$ and $\forall y \in W_m$ with $k, m < l$ we have $\varphi_{k,l}x +_l \varphi_{m,l}y \in W_l$. In particular, $\forall x, y \in W_{n-1}$ with $n \geq 2$ we have $\varphi_{n-1,n}x +_n \varphi_{n-1,n}y \in W_n$, $\varphi_{n-1,n}y +_n \varphi_{n-1,n}x \in W_n$ and $\varphi_{n-1,n}x +_n \varphi_{n-1,n}y = \varphi_{n-1,n}y +_n \varphi_{n-1,n}x$;

2. $\forall x \in W_k$ and $\forall y \in W_m$ with $k+m \le l$, we have $\varphi_{k,l}x \times_l \varphi_{m,l}y = \varphi_{m,l}y \times_l \varphi_{k,l}x \in W_l$. Moreover, $\varphi_{k,l}x \times_l \varphi_{m,l}y = \varphi_{k,r}x \times_r \varphi_{m,r}y$ for $r \ge l$. In this case, we call this multiplication standard or usual one and their results are the same. In particular,$\forall x, y \in W_{Ent[0.5n]}$ with $n \ge 2$ we have $\varphi_{Ent[0.5n],n}x \times_n \varphi_{Ent[0.5n],n}y \in W_n$, $\varphi_{Ent[0.5n],n}y \times_n \varphi_{Ent[0.5n],n}x \in W_n$ and $\varphi_{Ent[0.5n],n}x \times_n \varphi_{Ent[0.5n],n}y = \varphi_{Ent[0.5n],n}y \times_n \varphi_{Ent[0.5n],n}x$;

3. $\forall x \in W_k$, $\forall y \in W_l$, and $\forall z \in W_m$ with $m \ge l \ge k$ and $r \ge 2$, we have $(\varphi_{k,m+r}x +_{m+r} \varphi_{l,m+r}y) +_{m+r} \varphi_{m,m+r}z = \varphi_{k,m+r}x +_{m+r} (\varphi_{l,m+r}y +_{m+r} \varphi_{m,m+r}z) \in W_{m+r}$. In particular, $\forall x, y, z \in W_{n-2}$ with $n \ge 3$ we have $\varphi_{n-2,n}x +_n \varphi_{n-2,n}y \in W_n$, $(\varphi_{n-2,n}x +_n \varphi_{n-2,n}y) +_n \varphi_{n-2,n}z \in W_n$, $\varphi_{n-2,n}y +_n \varphi_{n-2,n}z \in W_n$, $\varphi_{n-2,n}x +_n (\varphi_{n-2,n}y +_n \varphi_{n-2,n}z) \in W_n$, and $(\varphi_{n-2,n}x +_n \varphi_{n-2,n}y) +_n \varphi_{n-2,n}z = \varphi_{n-2,n}x +_n (\varphi_{n-2,n}y +_n \varphi_{n-2,n}z)$;

4. $\forall x \in W_k$, $\forall y \in W_l$, and $\forall z \in W_m$ with $m \ge l \ge k$, we have $(\varphi_{k,k+l+m}x \times_{k+l+m} \varphi_{l,k+l+m}y) \times_{k+l+m} \varphi_{m,k+l+m}z = \varphi_{k,k+l+m}x \times_{k+l+m} (\varphi_{l,k+l+m}y \times_{k+l+m} \varphi_{m,k+l+m}z) \in W_{k+l+m}$. We also have, that for $r \ge k+l+m$, $(\varphi_{k,k+l+m}x \times_{k+l+m} \varphi_{l,k+l+m}y) \times_{k+l+m} \varphi_{m,k+l+m}z = (\varphi_{k,r}x \times_r \varphi_{l,r}y) \times_r \varphi_{m,r}z \in W_r$. In particular, $\forall x, y, z \in W_{Ent[0.25n]}$ with $n \ge 4$ we have $\varphi_{Ent[0.25n],n}x \times_n \varphi_{Ent[0.25n],n}y \in W_n$, $\left(\varphi_{Ent[0.25n],n}x \times_n \varphi_{Ent[0.25n],n}y\right) \times_n \varphi_{Ent[0.25n],n}z \in W_n$, $\varphi_{Ent[0.25n],n}y \times_n \varphi_{Ent[0.25n],n}z \in W_n$, $\varphi_{Ent[0.25n],n}x \times_n \left(\varphi_{Ent[0.25n],n}y \times_n \varphi_{Ent[0.25n],n}z\right) \in W_n$ and $(\varphi_{Ent[0.25n],n}x \times_n \varphi_{Ent[0.25n],n}y) \times_n \varphi_{Ent[0.25n],n}z = \varphi_{Ent[0.25n],n}x \times_n (\varphi_{Ent[0.25n],n}y \times_n \varphi_{Ent[0.25n],n}z)$;

5. $\forall x \in W_k$, $\forall y \in W_l$, and $\forall z \in W_m$ with $m \ge l \ge k$ and for $r \ge 1$ we have

$$(\varphi_{k,l+m+r}x +_{l+m+r} \varphi_{l,l+m+r}y) \times_{l+m+r} \varphi_{m,l+m+r}z =$$

$$= \varphi_{k,l+m+r}x \times_{l+m+r} \varphi_{m,l+m+r}z + \varphi_{l,l+m+r}y \times_{l+m+r} \varphi_{m,l+m+r}z \in W_{l+m+r}$$

In particular, $\forall x, y, z \in W_{Ent[0.3n]}$ with $n \geq 3$ we have $\varphi_{Ent[0.3n],n} y +_n \varphi_{Ent[0.3n],n} z \in W_n$, $\varphi_{Ent[0.3n],n} x \times_n \left(\varphi_{Ent[0.3n],n} y +_n \varphi_{Ent[0.3n],n} z\right) \in W_n$, $\varphi_{Ent[0.3n],n} x \times_n \varphi_{Ent[0.3n],n} y \in W_n$, $\varphi_{Ent[0.3n],n} x \times_n \varphi_{Ent[0.3n],n} z \in W_n$, and finally, $\varphi_{Ent[0.3n],n} x \times_n (\varphi_{Ent[0.3n],n} y +_n \varphi_{Ent[0.3n],n} z) = \varphi_{Ent[0.3n],n} x \times_n \varphi_{Ent[0.3n],n} y +_n \varphi_{Ent[0.3n],n} x \times_n \varphi_{Ent[0.3n],n} z$;

6. $\exists \quad 0 = 0.\underbrace{0...0}_{n} \in W_n \quad a +_n 0 = 0 +_n a = a \quad \forall a \in W_n$;

7. $\exists \quad 1 = 1.\underbrace{0...0}_{n} \in W_n \quad a \times_n 1 = 1 \times_n a = a \quad \forall a \in W_n$;

8. $\forall x \in W_n \quad \exists \quad -x \in W_n \quad x +_n (-x) = -x +_n x = 0$;

   Note, that for $n \geq 4$, $n - 1 > n - 2 \geq Ent\,[0.3n] \geq Ent\,[0.25n]$, hence $W_n \supset W_{n-1} \supset W_{n-2} \supset W_{Ent[0.3n]} \supset W_{Ent[0.25n]}$.

9. If $x, y \in W_n$ and $x +_n y \in W_n$ then $y +_n x \in W_n$ and $x +_n y = y +_n x$;

Here we provide some basic examples to illustrate what might happen whenever conditions 1-9 above are violated.

1. Additive associativity fails:$(x +_n y) +_n z \neq x +_n (y +_n z)$, e.g. let $10,\ 95,\ -35 \in W_2$, then $10 +_2 95 \notin W_2$, hence $(10 +_2 95) +_2 (-35) \notin W_2$, but $10 + ({}_2 95 +_2 (-35)) = 70 \in W_2$;

2. Multiplicative associativity fails:$(x \times_n y) \times_n z \neq x \times_n (y \times_n z)$, e.g. let 50.12, 0.85, and $0.61 \in W_2$, then $50.12 \times_2 0.85 = (50 + 0.1 + 0.02) \cdot (0.8 + 0.05) = 40 + 2.5 + 0.08 = 42.58$, and $(50.12 \times_2 0.85) \times_2 0.61 = (42 + 0.5 + 0.08) \cdot (0.6 + 0.01) = 25.2 + 0.42 + 0.3 = 25.65$, whereas $0.85 \times_2 0.61 = (0.8 + 0.05) \cdot (0.6 + 0.01) = 0.48$ and $50.12 \times_2 (0.85 \times_2 0.61) = (50 + 0.1 + 0.02) \cdot (0.4 + 0.08) = 20 + 4 + 0.04 = 24.04$;

3. Distributivity fails: $x \times_n (y +_n z) \neq x \times_n y +_n x \times_n z$, e.g. let $1.81,\ 0.74,\ 0.53 \in W_2$, then $0.74 +_2 0.53 = 1.27$ and $1.81 \times_2 (0.74 +_2 0.53) = (1 + 0.8 + 0.01) \cdot (1 + 0.2 + 0.07) = 1 + 0.2 + 0.07 + 0.8 + 0.16 + 0.01 = 2.24$, whereas $1.81 \times_2 0.74 = (1 + 0.8 + 0.01) \cdot (0.7 + 0.04) = 0.7 + 0.04 + 0.56 = 1.3$ and $1.81 \times_2 0.53 = (1 + 0.8 + 0.01) \cdot (0.5 + 0.03) = 0.5 + 0.03 + 0.4 = 0.93$, so that $1.81 \times_2 0.74 +_2 1.81 \times_2 0.53 = 2.23$;

4. Lack of the distribution law leads to the following results:

4a. The famous multiplication identities are invalid, e.g. $(x + y)^2 \neq x^2 + 2\,(xy) + y^2$.

We now have the following

THEOREM 2.4. *$P\,((a +_n b) \times_n (a +_n b) = (a \times_n a +_n 2 \times_n (a \times_n b)) +_n b \times_n b) < 1$, where P is the probability.* The proof of this theorem follows from the following (also see 4). Let $n = 2$. Then

1. The left hand side is $(1.32 +_2 2.43) \times_2 (1.32 +_2 2.43) = 3.75 \times_2 3.75 = 13.99$, while the right hand side is calculated in parts. First, $1.32 \times_2 1.32 = 1.73$; second, $2 \times_2 (1.32 \times_2 2.43) = 6.38$, and third $2.43 \times_2 2.43 = 5.88$. This means that $1.73 +_2 6.38) +_2 5.88 = 13.99$. I.e. the left hand side is indeed equal to the right hand side. However, observe the calculations in step 2.

2. The left hand side is $(1.32 +_2 2.79) \times_2 (1.32 +_2 2.79) = 4.11 \times_2 4.12 = 16.89$, while the right hand side is calculated in part as well. First, $1.32 \times_2 1.32 = 1.73$; second, $2 \times_2 (1.32 \times_2 2.79) = 7.28$, and third $2.79 \times_2 2.79 = 7.65$. This means that $1.73 +_2 7.28) +_2 7.65 = 16.66$. I.e. the left hand side is not equal to the right hand side.

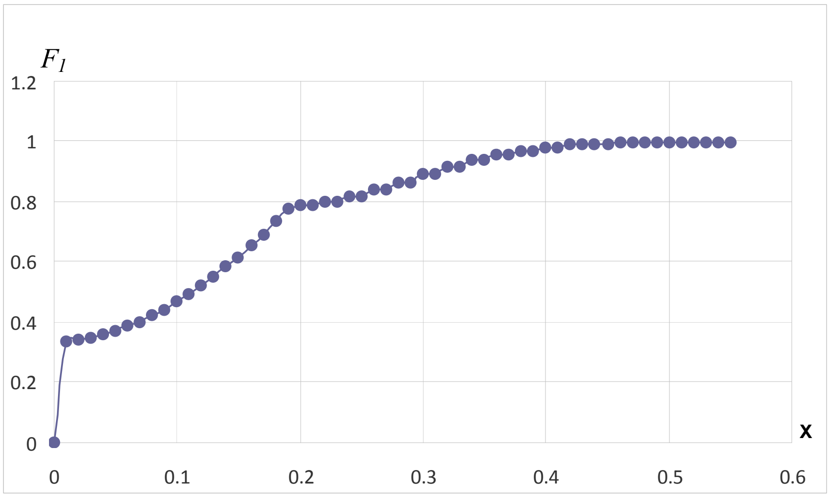


Figure 1. Graph of F_1.

In particular, for W_2, direct calculation shows that $P = 0.34$. Now, consider a random variable

$$\delta_1 = (a +_n b) \times_n (a +_n b) - ((a \times_n a +_n 2 \times_n (a \times_n b)) +_n (b \times_n b))$$

where $a, b \geq 0$, and δ_1 and all expressions on the right hand side are in W_n. Now, put $n = 2$. Then using direct calculations, we can build $F_1(x)$ - distribution function of δ_1, according to the following expression $F_1(x) = P(\delta_1 < x)$, where P is the probability. The graph of $F_1(x)$ is given by Figure 1.

General proof for W_n follows from the information below. If a, b are positive integers in W_n and $(a +_n b) \times_n (a +_n b) \in W_n$, then we have $\delta_1 = 0$. Consider now $a = 0.\underbrace{9 \ldots 9}_{n}$ and $b = 0.\underbrace{0 \ldots 08}_{n}$. Then $a +_n b = 1.\underbrace{0 \ldots 07}_{n}$ and $(a +_n b) \times_n (a +_n b) = 1.\underbrace{0 \ldots 07}_{n} \times_n 1.\underbrace{0 \ldots 07}_{n} = 1.\underbrace{0 \ldots 014}_{n}$, however, $a \times_n a < 1$, $b \times_n b = 0$, and $2 \times_n (a \times_n b) = 0$. Thus, $\delta_1 \neq 0$.

We now have another theorem, see 4 and 5.

THEOREM 2.5. *$P(c \times_n (a +_n b) = c \times_n a +_n c \times_n b) < 1$, where P is the probability.* The proof of this theorem follows from the following. Let $n = 2$. Then

1. The left hand side is $2 \times_2 (3 +_2 6) = 2 \times_2 9 = 18$, and the right hand side is calculated in parts. First, $2 \times_2 3 = 6$, then $2 \times_2 6 = 12$ and $6 +_2 12 = 18$ I.e. the left hand side is indeed equal to the right hand side. However, observe the calculations in step 2.

2. The left hand side is $2.41 \times_2 (3.14 +_2 0.58) = 2.41 \times_2 3.72 = 8.95$, and the right hand side is calculated in parts. First, $2.41 \times_2 3.14 = 7.55$, then $2.41 \times_2 0.58 = 1.36$ and $7.55 +_2 1.36 = 8.91$ I.e. the left hand side is not equal to the right hand side.

In particular, for W_2, direct calculation shows that $P = 0.34$. Now, consider a random variable

$$\delta_2 = c \times_n (a +_n b) -_n (c \times_n a +_n c \times_n b)$$

where $a, b, c \geq 0$, and δ_2 and all expressions on the right hand side are in W_n. Now, put $n = 2$. Then using direct calculations, we can build $F_2(x)$ - distribution function of δ_2, according to the following expression $F_2(x) = P(\delta_2 < x)$, where P is the probability. The graph of $F_2(x)$ is given by Figure 2.

General proof for W_n follows from the information below. If a, b, c are positive integers in W_n and $a \times_n (b \times_n c) \in W_n$, then we have $\delta_2 = 0$. Consider now $a = 2$, $b = 0.\underbrace{9\ldots9}_{n}$ and $c = 0.\underbrace{0\ldots01}_{n}$. Then $b \times_n c = 0$, $a \times_n (b \times_n c) = 0$, $a \times_n b = 1.\underbrace{9\ldots98}_{n}$, and $(a \times_n b) \times_n c = 0.\underbrace{0\ldots01}_{n}$. Thus, $\delta_2 \neq 0$.

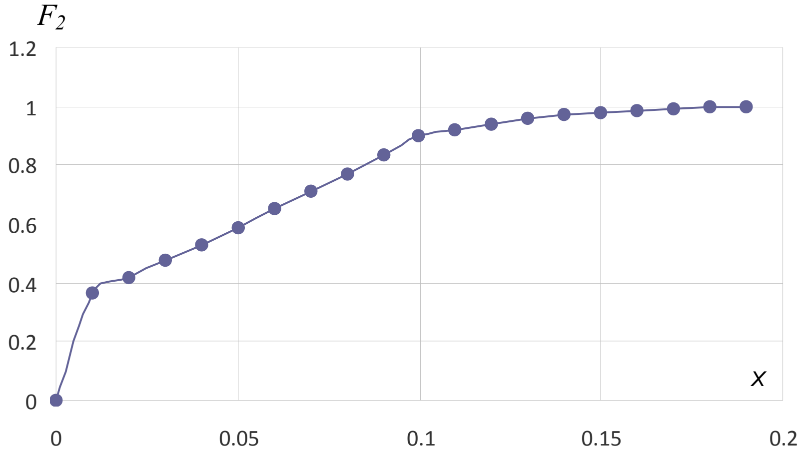


Figure 2. Graph of F_2.

4b. The statement "$x|y$ and $x|z \quad \Rightarrow \quad x|\,(y+z)$" is false. Here $x|y \quad \Leftrightarrow \exists r x \times_n r = y$. Assume that $x|y$ and $x|z$, what we want to show is equivalent to showing that $y + z \neq x \times_n (r_1 +_n r_2)$ for some x, y, z, r_1 and r_2. Let $x = 0.17$, $r_1 = 0.85$, $r_1 = 0.63$, $y = 0.17 \times_2 0.85 = 0.(0.1+0.07)\cdot(0.8+0.05) = 0.08$ and $z = 0.17 \times_2 0.63 = 0.(0.1+0.07)\cdot(0.6+0.03) = 0.06$. Then $y + z = 0.14$, but $r_1 + r_2 = 1.48$ and $0.17 \times_2 1.48 = (0.1 + 0.07) \cdot (1 + 0.4 + 0.08) = 0.1 + 0.04 + 0.07 = 0.21$. In fact, $xy+z$. This is because if we let $0.17 \times_2 0.9 = (0.1 + 0.07) \cdot 0.9 = 0.09 < 0.14$ and $0.17 \times_2 0.99 = (0.1 + 0.07) \cdot (0.9 + 0.09) = 0.09 < 0.14$, but $0.17 \times_2 1 = 0.17 > 0.14$.

5. Multiplicative inverses do not necessarily exist, or if they do, they are not necessarily unique in W_n. Here are some examples: let $2 \in W_n$, then $0.5 \in W_2$is the unique inverse of 2 for any W_n. On the other hand, 3 will not have an inverse in any W_n. Now, let $2^{-1} = 0.5$, then $(0.5)^{-1}$is actually the following set $\{2, 2.01, 2.02, 2.03, 2.04, 2.05, 2.06, 2.07, 2.08, 2.09\} \in W_2$.

Therefore, $\left(2^{-1}\right)^{-1}$ is not necessarily 2, hence all we can claim is that if x^{-1} exists, then $x \in \left\{\left(x^{-1}\right)^{-1}\right\}$. Further, if an inverse of an element exists in W_n, it does not necessarily exist in W_m for $m \neq n$, independent of the order of $m$and n, e.g. if $0.91 \in W_2$, then $(0.91)^{-1} = \{1.1, 1.11, 1.12, 1.13, 1.14, 1.15, 1.16, 1.17, 1.18, 1.19\} \in W_2$, but $(0.91)^{-1} \notin W_4$, on the other hand, $16^{-1} = 0.0625 \in W_4$, but $16^{-1} \notin W_2$.

6. Square roots do not necessarily exist. Some examples are, if $4 \in W_n$, then $\sqrt{4} = 2$ for any n and $\sqrt{3}$ does not exist in $n = 2$. To show that, consider $1.75 \times_2 1.75 = (1 + 0.7 + 0.05) \cdot (1 + 0.7 + 0.05) = 1 + 0.7 + 0.05 + 0.7 + 0.49 + 0.05 = 2.99$ and

$$1.76 \times_2 1.76 = (1 + 0.7 + 0.06) \cdot (1 + 0.7 + 0.06) = 1 + 0.7 + 0.06 + 0.7 + 0.49 + 0.06 = 3.01.$$

Further, if a square root of an element exists in W_n, it does not necessarily exist in W_m for $m \neq n$, independent of the order of m and n, e.g. $\sqrt{2} = 1.42 \in W_2$, since $1.42 \times_2 1.42 = (1 + 0.4 + 0.02) \cdot (1 + 0.4 + 0.02) = 1 + 0.4 + 0.02 + 0.4 + 0.16 + 0.02 = 2$, but $\sqrt{2} \notin V_4$,

since $1.4143 \times_4 1.4143 = (1 + 0.4 + 0.01 + 0.004 + 0.0003) \cdot (1 + 0.4 + 0.01 + 0.004 + 0.0003) = 1.9999$

and $1.4144 \times_4 1.4144 = 2.0001$. Also, $\sqrt{1.01} = 1.005 \in W_4$, since $1.005 \times_4 1.005 = (1 + 0.005) \cdot (1 + 0.005) = 1 + 0.005 + 0.005 = 1.01$, but $\sqrt{1.01} \notin W_2$, since $1 \times_2 1 = 1$ and $1.01 \times_2 1.01 = (1 + 0.01) \cdot (1 + 0.01) = 1 + 0.01 + 0.01 = 1.02$.

Next, some basic theorems can be stated for W_n:

1. Any W_n has zero divisors: $0.\underbrace{0...0}_{n-1}1 \times_n 0.\underbrace{0...0}_{n-1}1 = 0$;
2. If $p \in W_n$ with $p \neq 2, 5$ a prime in the usual sense, then $p^{-1} \notin W_n$ for any W_n;

3. $\forall x, y \in W_n$ with $x, y \geq 0$, $x - y \in W_n$.

4. If $x, y, t, u \in W_n$ and $x \geq t \geq 0$ and $y \geq u \geq 0$, then $x \times_n y \geq t \times_n u$

5. If given $a \in W_n$ such that there is a unique $a^{-1} \in W_n$, then $|a| \geq 1$;

6. If $|a| < 1$ and a^{-1} exists, then $|\{a^{-1}\}| > 1$;

7. If $|\{a^{-1}\}| > 1$, then $|a| < 1$.

THEOREM 2.6. *Let* $\delta_3 = c \times_n (a \times_n b) -_n (c \times_n a) \times_n b$. *Then* $0 < P(\delta_3 = 0) < 1$, *where* P *is the probability.*

The proof of Theorem 2.6 follows from the following (see 5). Let $n = 2$. Then

1. The left hand side is $2 \times_2 (3 \times_2 6) = 2 \times_2 18 = 36$, and the right hand side is calculated in parts. First, $2 \times_2 3 = 6$, then $6 \times_2 6 = 36$. I.e. the left hand side is indeed equal to the right hand side. However, observe the calculations in step 2.

2. The left hand side is $2.41 \times_2 (3.14 \times_2 0.58) = 2.41 \times_2 1.79 = 4.27$, and the right hand side is calculated in parts. First, $2.41 \times_2 3.14 = 7.55$, then $7.55 \times_2 0.58 = 4.31$. I.e. the left hand side is not equal to the right hand side. In particular, for W_2, direct calculation shows that $P = 0.07$. Now, consider a random variable $\delta_3 = c \times_n (a \times_n b) -_n (c \times_n a) \times_n b$, where $a, b, c \geq 0$, and δ_3 and all expressions on the right hand side are in W_n. If we put $n = 2$, then using direct calculations, we can build $F_3(x)$ - distribution function of δ_3, according to the following expression $F_3(x) = P(\delta_3 < x)$, where P is the probability. The graph of $F_3(x)$ is given by the following figure. General proof for W_n follows from the below. If

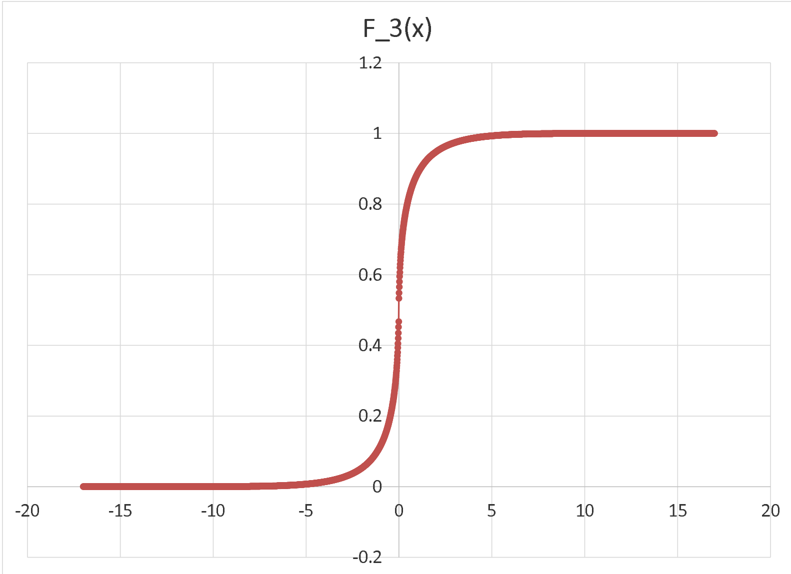


Figure 3. Graph of F_3.

a, b, c are positive integers in W_n and $c \times_n (a \times_n b) \in W_n$, then we have $\delta_3 = 0$. Consider now $c = 2$, $a = 0.\underbrace{9\ldots99}_{n}$ and $b = 0.\underbrace{0\ldots01}_{n}$. Then

$$\delta_3 = 2 \times_n (0.9\ldots99 \times_n 0.0\ldots01) -_n (2 \times_n 0.9\ldots99) \times_n 0.0\ldots01$$

$$= 0 -_n 0.0\ldots01 = -0.0\ldots01 \neq 0$$

We will now introduce the complexification of W_n, the set CW_n, with $n = 2, 3, \ldots$. We call CW_n the set of all formal sums $z = a + ib$ with the following arithmetic:

$$(a_1 + ib_1) +_n (a_2 + ib_2) = (a_1 +_n a_2) + i(b_1 +_n b_2)$$

and

$$(a_1 + ib_1) \times_n (a_2 + ib_2) = (a_1 \times_n a_2 -_n b_1 \times_n b_2) + i(a_1 \times_n b_2 +_n a_2 \times_n b_1)$$

We have the same properties of CW_n as we have in W_n. In particular,

$$z_1 +_n z_2 = z_2 +_n z_1$$

$$z_1 \times_n z_2 = z_2 \times_n z_1$$

We also have zero-divisors in CW_n and associativity of summation and multiplication as well as distributivity become invalid sometimes.

2.3 Algebra

This section deals with algebraic properties of the sets W_n and how they illustrate the fact of relativity of mathematics. We begin with the most basic algebraic equation $a \times_n x = b$. Now, due to the rules of arithmetic in any W_n we have the following cases. Suppose $a \in W_n$ such that $a^{-1} \notin W_n$, then any of the following can occur: we can have a unique solution, e.g. $3 \in W_2$, $3^{-1} \notin W_2$, and $x = 1$ is the unique solution of $3 \times_2 x = 3$; many solutions, e.g. $0.3 \in W_2$, $0.3^{-1} \notin W_2$, and $x = 0.1, 0.11, ..., 0.19$ are the solutions of $0.3 \times_2 x = 0.03$; and no solutions, e.g. $3 \in W_2$, $3^{-1} \notin W_2$, and there is no solution to $3 \times_2 x = 1$. The next case is when there is a unique inverse a^{-1} for $a \in W_n$, then we have the following fact: $a \times_n x = b$ either has a unique solution or no solutions. That the equation has many solutions does not occur here. To see this, first note, that a unique inverse cannot exist if $|a| < 1$. Now, write the equation as $a_0.a_1...a_n \times_n x_0.x_1...x_n = b$ with $a_0 \neq 0$ and assume a solution exists. Then if we vary x_n between 0 and 9 the $a_0 \cdot 0.\underbrace{0...0}_{n-1} x_n$ term of the product will also vary, thus changing the product and invalidating the equality, hence the solution must be unique. Finally, we consider the case where $|\{a^{-1}\}| > 1$. The following is then true: $a \times_n x = b$ has either many solutions or no solutions. To see this, write $a_0.a_1...a_n \times_n x_0.x_1...x_n = b$ and assume that there is a solution. Now, note that if we vary x_n between 0 and 9 the term $0.\underbrace{0...0}_{n-2} a_{n-1} \cdot 0.\underbrace{0...0}_{n-1} x_n$of the product is irrelevant since, by definition, it drops off and we get many solutions.

Now we will show the independence of existence of solutions of the equation $a \times_n x = b$ by varying n. The cases that arise are as follows: if there exists a unique solution in W_n, that does not necessarily imply the existence of a solution in W_m for $m \neq n$. However, if there are many solutions to an equation in W_n, there will be the same number of solutions in W_m, but not necessarily the same ones. Here are some examples: $2 \times_2 x = 0.01$ has no solution, but $2 \times_4 x = 0.01$ has a unique solution$x = 0.005$. Both $3 \times_2 x = 18$ and $3 \times_4 x = 18$ have a unique solution$x = 6$. The equation$0.1 \times_2 x = 0.12$ has 10 solutions $\{1.2, 1.21, ..., 1.29\}$ and $0.1 \times_4 x = 0.12$ also has 10 solutions,$\{1.2, 1.2001, ..., 1.2009\}$. Note, that the solutions are different. Also, notice the two equations $0.1 \times_2 x = 0.12$ and $1 \times_2 x = 1.2 \Leftrightarrow x = 1.2$ are not equivalent due to different number of solutions.

We now consider systems of linear equations. Let us start with a special case. We know that in W_2, $2^{-1} = 0.5$ and $0.5^{-1} = \{2, 2.01, ..., 2.09\}$, then the system

$$\begin{cases} 2 \times_2 x = 0.32 \\ 2.01 \times_2 x = 0.32 \\ ... \\ 2.09 \times_2 x = 0.32 \end{cases}$$

has a unique solution $x = 0.16$, moreover each equation in the system also has $x = 0.16$ as a unique solution. In fact, we have the following theorem: the system $a_i \times_n x = b$ such that $a_i \in \{a^{-1}\}$for some $a \in W_n$ either has no solution (in this case each equation has no solution) or has a unique solution (in this case, each equation has the same unique solution).

Next we consider systems of two linear equations with two unknowns, their solutions in W_n and W_m for $n \neq m$, and also show that systems can be nonequivalent after elementary row operations. For example, consider

$$\begin{cases} 0.14 \times_2 x +_2 0.23 \times_2 y = 0.22 \\ 0.61 \times_2 x +_2 0.43 \times_2 y = 0.76 \end{cases}$$

then, for example, $x = 0.83$ and $y = 0.79$ is a solution, and therefore, there are actually 100 solutions in W_2:

$$\left\{ (0.8, 0.7) \quad (0.8, 0.71) \quad ... \quad (0.8, 0.79) \right\}$$
$$\left\{ (0.81, 0.7) \quad (0.81, 0.71) \quad ... \quad (0.81, 0.79) \right\}$$
$$\cdots$$
$$\left\{ (0.89, 0.7) \quad (0.89, 0.71) \quad ... \quad (0.89, 0.79) \right\}.$$

Now, consider

$$\begin{cases} 0.14 \times_4 x +_4 0.23 \times_4 y = 0.22 \\ 0.61 \times_4 x +_4 0.43 \times_4 y = 0.76 \end{cases}$$

then an easy computation shows that any solution of the W_2 system is not a solution in W_4. To see this, take the minimal solution from W_2, then $0.14 \times_4 0.8 +_4 0.23 \times_4 0.7 = 0.273$ and obviously any other solution will produce a larger result, hence cannot be a solution of this system. Now, by computing the solution to the system (using regular real numbers), we get numbers that in W_2 are $x = 1$ and $y = 0.35$, then by incrementing these values by 0.01, we see that there can be no solutions in W_4. On the other hand, consider

$$\begin{cases} 10 \times_4 x +_4 20 \times_4 y = 0.07 \\ 20 \times_4 x +_4 10 \times_4 y = 0.05 \end{cases}$$

This system has a (in fact, unique) solution $x = 0.0010$, $y = 0.0030$, whereas the system

$$\begin{cases} 10 \times_2 x +_2 20 \times_2 y = 0.07 \\ 20 \times_2 x +_2 10 \times_2 y = 0.05 \end{cases}$$

has no solution. Thus, the order of m and n has no influence on solutions. Other situations are also possible. For example,

$$\begin{cases} 1 \times_n x +_n 1 \times_n y = 3 \\ 2 \times_n x +_n 1 \times_n y = 4 \end{cases}$$

has a solution ($x = 1$, $y = 2$) for $n = 2, 4$, whereas the system

$$\begin{cases} 1 \times_n x +_n 1 \times_n y = 3 \\ 2 \times_n x +_n 2 \times_n y = 5 \end{cases}$$

has solutions for neither values of n.

Let us consider the problem of determining equivalency between systems and their elementary transformation (via row operations). Given

$$\begin{cases} 0.14 \times_2 x +_2 0.23 \times_2 y = 0.22 \;\; (1) \\ 0.61 \times_2 x +_2 0.43 \times_2 y = 0.76 \;\; (2) \end{cases}$$

consider

$$\begin{cases} 0.14 \times_2 x +_2 0.23 \times_2 y = 0.22 \;\; (1) \\ 0.75 \times_2 x +_2 0.66 \times_2 y = 0.98 \;\; (1) + (2) \end{cases}$$

Now, ignore the possibility of noncommutativity and pick any solution, e.g. $(0.8, 0.7)$, of the first system and plug it into the second system. An easy computation shows that the solution does not satisfy the $(1) + (2)$. In fact, no other solution will satisfy it, hence the two systems are nonequivalent. Next, consider

$$\begin{cases} 0.14 \times_2 x +_2 0.23 \times_2 y = 0.22 \;\; (1) \\ 1.22 \times_2 x +_2 0.86 \times_2 y = 1.52 \;\; (2) \cdot 2 \end{cases}$$

Again, ignore the possibility of noncommutativity and pick a solution, e.g. $(0.81, 0.71)$, to the system with rows (1) and (2), then it easy to see that it does not satisfy the system with rows

(1) and (2) $\cdot$ 2. In fact, all other solutions except $(0.8, 0.7)$ do not satisfy this system, hence, again, the systems are not equivalent. Similar analysis shows that the system

$$\begin{cases} 0.14 \times_2 x +_2 0.23 \times_2 y = 0.22 \ \ (1) \\ 6.24 \times_2 x +_2 4.53 \times_2 y = 7.82 \ \ (1) + (2) \cdot 10 \end{cases}$$

is not equivalent to the original system. Therefore, the elementary row operations produce nonequivalent systems of equations.

Here is another example. Consider the following system:

$$\begin{cases} 1 \times_n x +_n 1 \times_n y = 1 \\ 0.11 \times_n x +_n 0.37 \times_n y = 0.44 \end{cases}$$

Now, no matter that $\begin{vmatrix} 1 & 1 \\ 0.11 & 0.37 \end{vmatrix} \neq 0$ for any $n \geq 2$, we have, for example, that there are solutions for $n = 3, 5, 6, 7, 9$, and yet no solutions for $n = 2, 4, 8$.

We move now to the Cartesian product $\underbrace{W_n \times ... \times W_n}_{k}$. This is just the standard Cartesian product, with the natural addition and constant multiplication: $(x_1, ..., x_k) +_n (y_1, ..., y_k) = (x_1 +_n y_1, ..., x_k +_n y_k)$ and $\alpha \times_n (x_1, ..., x_k) = (\alpha \times_n x_1, ..., \alpha \times_n x_k)$ for $x_1, ..., x_k, y_1, ..., y_n, \alpha \in W_n$. Now, in order for this product to make sense to a W_n-observer, it must be such that $1 \leq k \leq \underbrace{9...9}_{n}$. We can work with the standard notions when $k = 2$ - plane, and $k = 3$ - space. The classical axioms of a linear space are also valid here whenever $x_1, ..., x_k, y_1, ..., y_n, \alpha \in W_{Ent[0.3n]}$, but in general, these properties are not valid due to lack of associativity and distributivity.

Now what is left is to define $\dim W_n$. We introduce two alternative definitions. We first define $\dim_1 W_n = \max s$, where s is the index of $u_0, u_1, ..., u_s$ such that $u_0 \in W_n$, $u_1 \in W_n \setminus \{W_n \times_n u_0\}$ such that $\{W_n \times_n u_0\} \not\subset \{W_n \times_n u_1\}$;

$u_2 \in W_n \backslash (\{W_n \times_n u_0\} +_n \{W_n \times_n u_1\})$ such that $\{W_n \times_n u_0\} \not\subset \{W_n \times_n u_2\}$ and $\{W_n \times_n u_1\} \not\subset \{W_n \times_n u_2\}$;

...

$u_k \in W_n \backslash (...((\{W_n \times_n u_0\} +_n \{W_n \times_n u_1\}) +_n \{W_n \times_n u_1\}) +_n ... +_n \{W_n \times_n u_{k-1}\})$ such that $\{W_n \times_n u_0\}, ..., \{W_n \times_n u_{k-1}\} \not\subset \{W_n \times_n u_k\}$ and finally, $W_n \backslash (\ldots((\{W_n \times_n u_0\} +_n \{W_n \times_n u_1\}\}) +_n \{W_n \times_n u_1\}\}) +_n ... +_n \{W_n \times_n u_{s-1}\}\}) \neq \emptyset$, but $W_n \backslash (\ldots((\{W_n \times_n u_0\} +_n \{W_n \times_n u_1\}\}) +_n \{W_n \times_n u_1\}\}) +_n ... +_n \{W_n \times_n u_s\}\}) = \emptyset$.

The second dimension, $\dim_2 W_n = \max s$ where s is the index of $u_0, u_1, ..., u_s$ such that $u_0, u_1, ..., u_s \in W_n$ and $\{W_n \times_n u_i\} \not\subset \{W_n \times_n u_j\}$ for $i < j$ and $i = 0, ..., s-1$, and $j = 1, ..., s$ and $W_n \backslash (...((\{W_n \times_n u_0\} +_n \{W_n \times_n u_1\}) +_n \{W_n \times_n u_1\}) +_n ... +_n \{W_n \times_n u_{s-1}\}) \neq \emptyset$, but $W_n \backslash (...((\{W_n \times_n u_0\} +_n \{W_n \times_n u_1\}) +_n \{W_n \times_n u_1\}) +_n ... +_n \{W_n \times_n u_s\}) = \emptyset$.

From the point of view of an observer with a higher level of thickness, we have the following theorem: $\dim_i \underbrace{W_n \times ... \times W_n}_{k} = (\dim_i W_n)^k$ for $i = 1, 2$. Now, the relationship between the two definitions can be expressed in the following theorem: $\dim_2 W_n \geq \dim_1 W_n$.

Here is a useful result when dealing with W_2: $\dim_1 W_2 \geq 7$. To show equality, consider the set of elements $\{99.99, 99.98, 99.97, 99.95, 99.92, 99.90, 99.53\}$, we will show that this set spans W_2. Consider the following set

$$A = \{W_2 \times_2 99.99\} \cap \{W_2 \times_2 99.98\} \cap \{W_2 \times_2 99.97\} \cap \{W_2 \times_2 99.95\} \cap \{W_2 \times_2 99.92\} \cap$$
$$\cap \{W_2 \times_2 99.90\} \cap \{W_2 \times_2 99.53\}$$

Now, this set has 199 points, moreover $\{W_2 \times_2 99.99\} \backslash A = \{\pm 99.99\}$, $\{W_2 \times_2 99.98\} \backslash A = \{\pm 99.98\}$, $\{W_2 \times_2 99.97\} \backslash A = \{\pm 99.97\}$, $\{W_2 \times_2 99.95\} \backslash A = \{\pm 99.95\}$, $\{W_2 \times_2 99.92\} \backslash A =$

$\{\pm 99.92\}$,$\{W_2 \times_2 99.90\} \setminus A = \{\pm 99.90\}$ and $\{W_2 \times_2 99.53\} \setminus A = \{\pm 99.53\}$. Finally, to see that $W_2 = ((((((\{W_2 \times_2 99.99\} +_2 \{W_2 \times_2 99.98\}) +_2 \{W_2 \times_2 99.97\}) +_2 \{W_2 \times_2 99.95\}) +_2 \{W_2 \times_2 99.92\}) +_2 \{W_2 \times_2 99.90\}) +_2 \{W_2 \times_2 99.53\})$.

We can also have the following cases occur: $\{W_2 \times_2 98.99\} \cap \{W_2 \times_2 99.01\} = \{0\}$ so that we have two lines contained in W_2 intersecting only at zero. Also, we have the following theorem $W_2 = ((\{W_2 \times_2 99.01\} +_2 \{W_2 \times_2 98.99\}) +_2 \{W_2 \times_2 95.51\})$, moreover these three lines intersect only at zero.

Now we can consider the plane $W_2 \times_2 98.99 +_2 W_2 \times_2 0.01$ that lies entirely on the line $W_2 \times_2 1$. Note, that $W_2 \times_2 0.01 = \{0, \pm 0.01, ..., \pm 0.99\}$ and we can show that $W_2 \times_2 98.99 +_2 W_2 \times_2 0.01$ actual equals W_2, i.e. this plane coincides with the line.

Also we have that $W_2 \times_2 98.99 \cap W_2 \times_2 99.01 = \{0\}$, i.e. the space $W_2 \times_2 98.99 +_2 W_2 \times_2 99.01$ is generated by two intersecting (only at zero) systems of collinear vectors. Now, take $98.03 \in W_2 \times_2 98.99 +_2 W_2 \times_2 99.01 = B$ and consider $W_2 \times_2 98.03 \cap B$. Also $|W_2 \times_2 98.03 \cap B| = 31$ and hence $W_2 = ((W_2 \times_2 98.99 +_2 W_2 \times_2 99.01) +_2 W_2 \times_2 98.03)$. These results follow using direct calculations.

2.4 Geometry

In previous section we have defined a $k-$fold Cartesian product of W_n's. In particular, $W_n \times W_n$ is a plane and $W_n \times W_n \times W_n$ is a space. Now, a line in the plane will be defined to be

$$\{(x, y) \in W_n \times W_n | a \times_n x +_n b \times_n y +_n c = 0 \text{ for some } a, b, c \in W_n\}$$

and a plane in a space will be defined to be

$$\{(x, y, z) \in W_n \times W_n \times W_n | a \times_n x +_n b \times_n y +_n c \times_n z +_n d = 0 \text{ for some } a, b, c, d \in W_n\}$$

In particular, for $u \in W_n$ we have $\{W_n \times_n u\}$ - a line on a plane and simultaneously, when viewed as $y = x \times_n u$, a line on a line $\{W_n \times_n 1\}$. Also, for $u_1, u_2 \in W_n$ we have the plane $\{W_n \times_n u_1\} +_n \{W_n \times_n u_2\}$ that lies in space, but when viewed as $u_1 \times_2 x + u_2 \times_2 y = z$, it is a plane on the line $\{W_n \times_n 1\}$. In fact, we can also have a space $\{W_n \times_n u_1\} +_n \{W_n \times_n u_2\} +_n \{W_n \times_n u_3\}$ on that same line, etc. Also, as stated in previous sections, we can have a line containing two lines that intersect only at zero.

Now, let us consider intersection of lines on the plane $W_2 \times W_2$. Here are a few examples of how two lines that intersect in the usual sense actually intersect at no, one, two, ten and even a hundred points. For no intersection, consider

$$\begin{cases} 0.08 \times_2 x +_2 0.78 \times_2 y +_2 0.09 = 0 \\ -0.47 \times_2 x -_2 0.75 \times_2 y -_2 0.38 = 0 \end{cases}$$

For one point

$$\begin{cases} 0.59 \times_2 x +_2 0.79 \times_2 y +_2 0.59 = 0 \\ 1.00 \times_2 x +_2 1.00 \times_2 y +_2 0.41 = 0 \end{cases}$$

For two points

$$\begin{cases} 0.31 \times_2 x +_2 1.00 \times_2 y +_2 0.63 = 0 \\ 1.00 \times_2 x +_2 0.34 \times_2 y +_2 0.91 = 0 \end{cases}$$

For ten points

$$\begin{cases} 0.30 \times_2 x +_2 1.00 \times_2 y +_2 0.53 = 0 \\ 0.32 \times_2 x +_2 0.28 \times_2 y +_2 0.74 = 0 \end{cases}$$

Finally, for 100 points in the intersection, consider

$$\begin{cases} 0.14 \times_2 x +_2 0.23 \times_2 y -_2 0.22 = 0 \\ 0.61 \times_2 x +_2 0.43 \times_2 y -_2 0.76 = 0 \end{cases}$$

For a visual illustration of each, see Figures below.

Now, we can also have different situations occur depending on the observers. We can have two given lines intersecting in a plane for any W_n, not intersecting for any W_n, or intersect for some W_n, but not intersect for W_m such that $m \neq n$. In general, given 2-fold Cartesian products of $W_{n_1}, \ldots, W_{n_k}$ with $n_1 < ... < n_k$ there exists two lines with coefficients in W_{n_1}, such that they intersect for a given W_{n_i} and do not intersect for the others. Moreover, for any subset $\{n_i\}$ of $\{n_1, ..., n_k\}$, there exits two of lines, which intersect in $W_{\{n_i\}}$, but not in $\{n_1, ..., n_k\} \setminus \{n_i\}$.

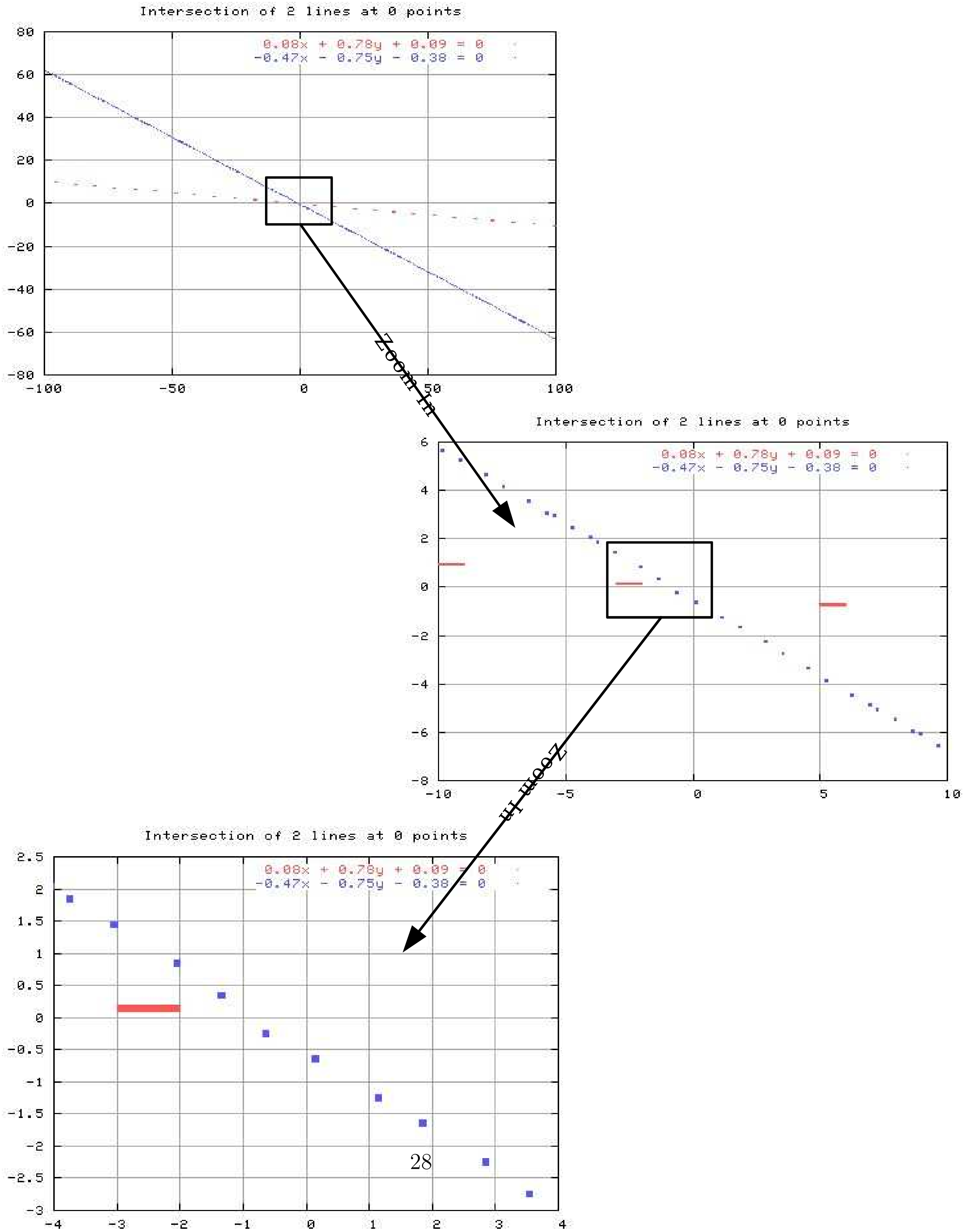
Here are some examples. Let

$$\begin{cases} y = x +_n 2 \\ y = 4 \times_n x +_n 1 \end{cases}$$

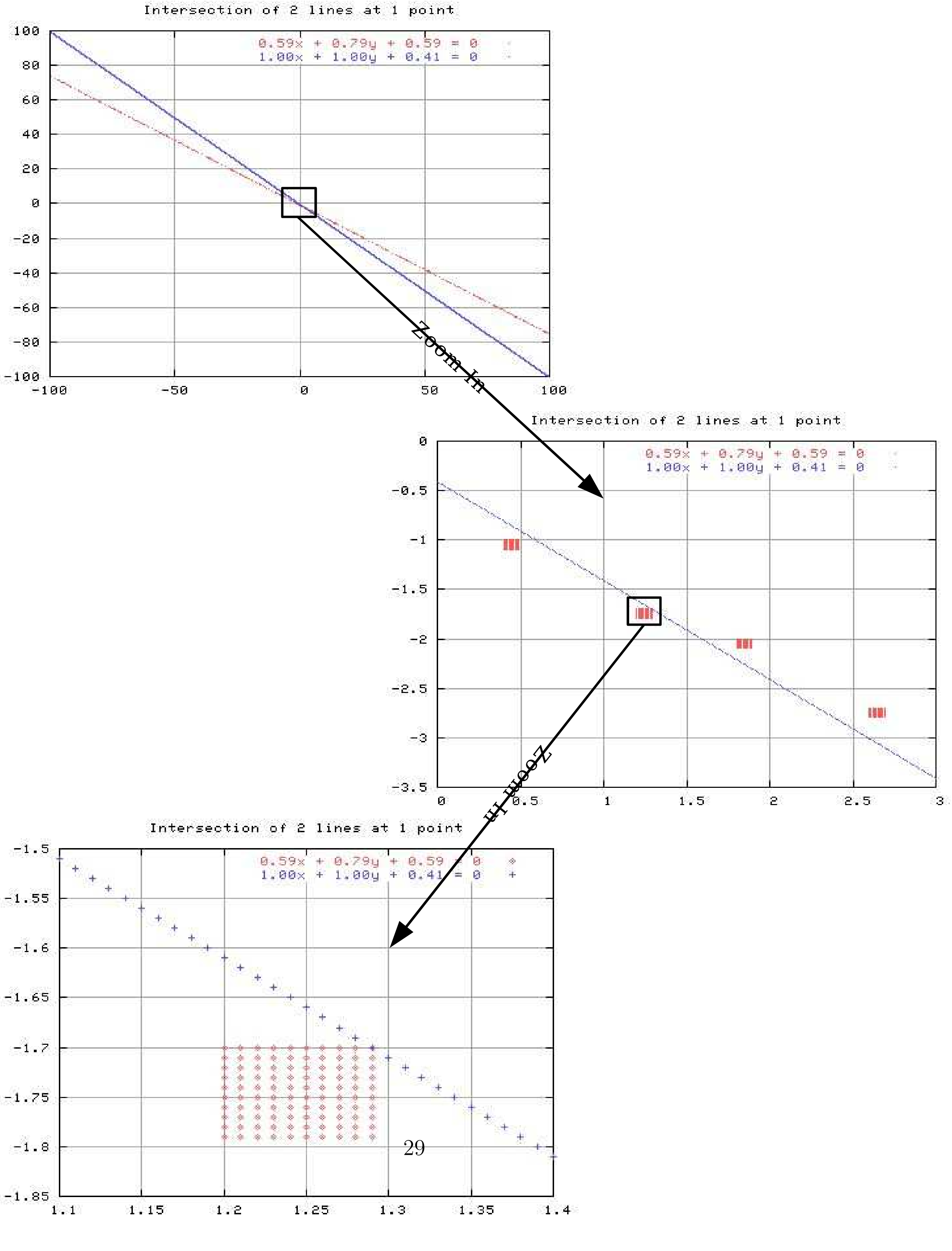
be the two lines, then solving this system, we get $1 = 3 \times_n x$ which has no solution for any n. Next, consider

$$\begin{cases} y = x +_n 1 \\ y = 3 \times_n x \end{cases}$$

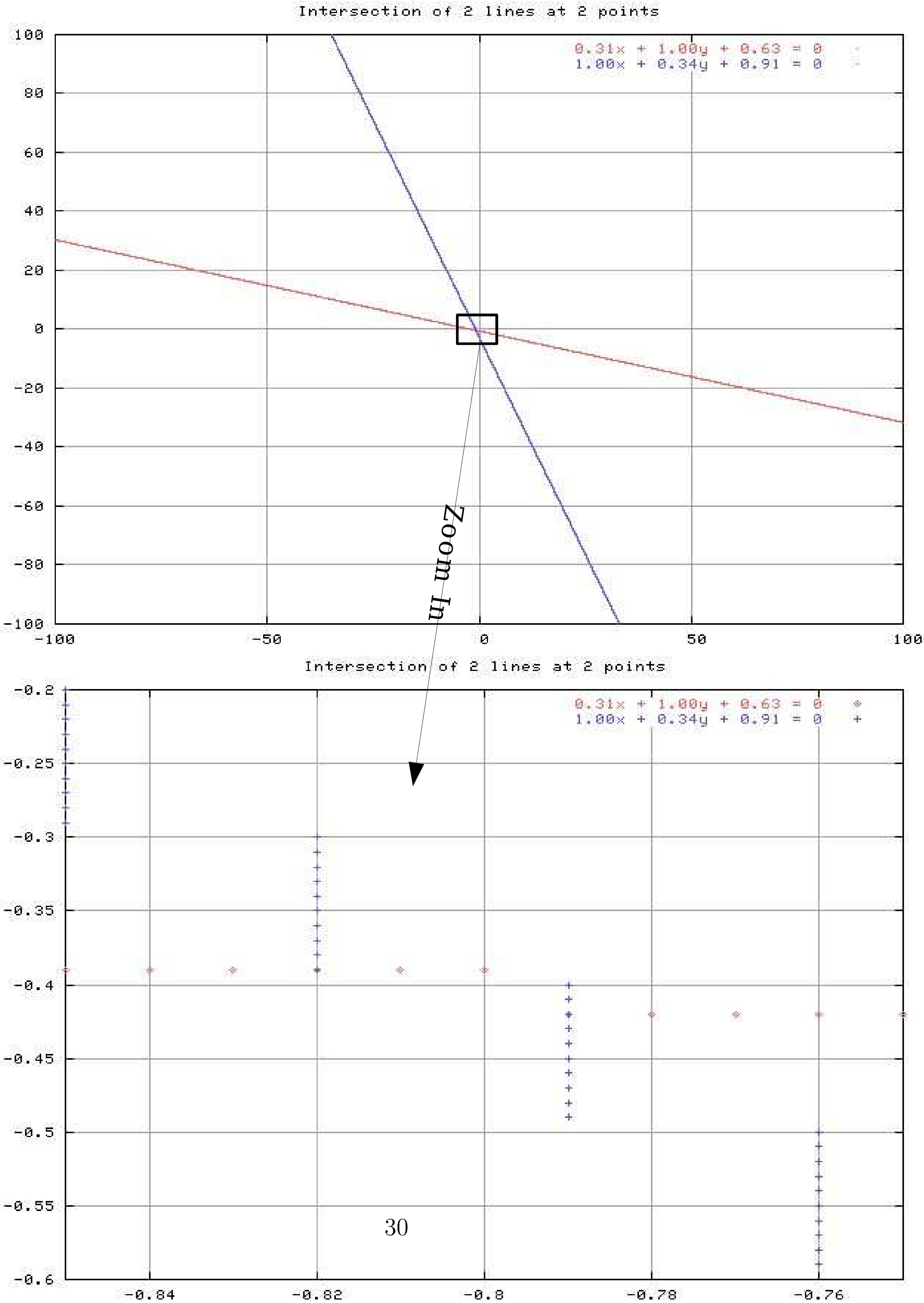
Intersection of 2 lines at 0 points



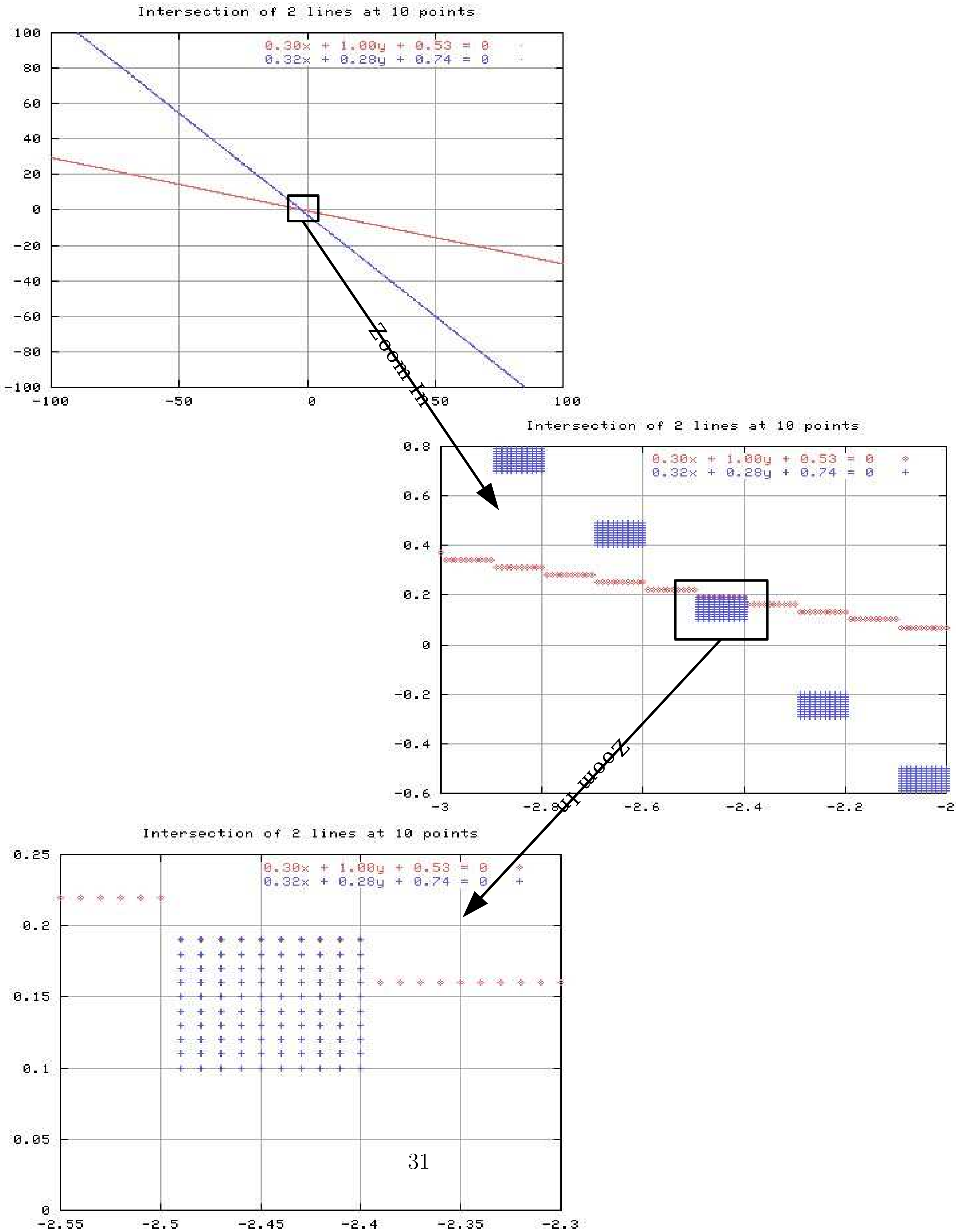
Intersection of 2 lines at 1 point



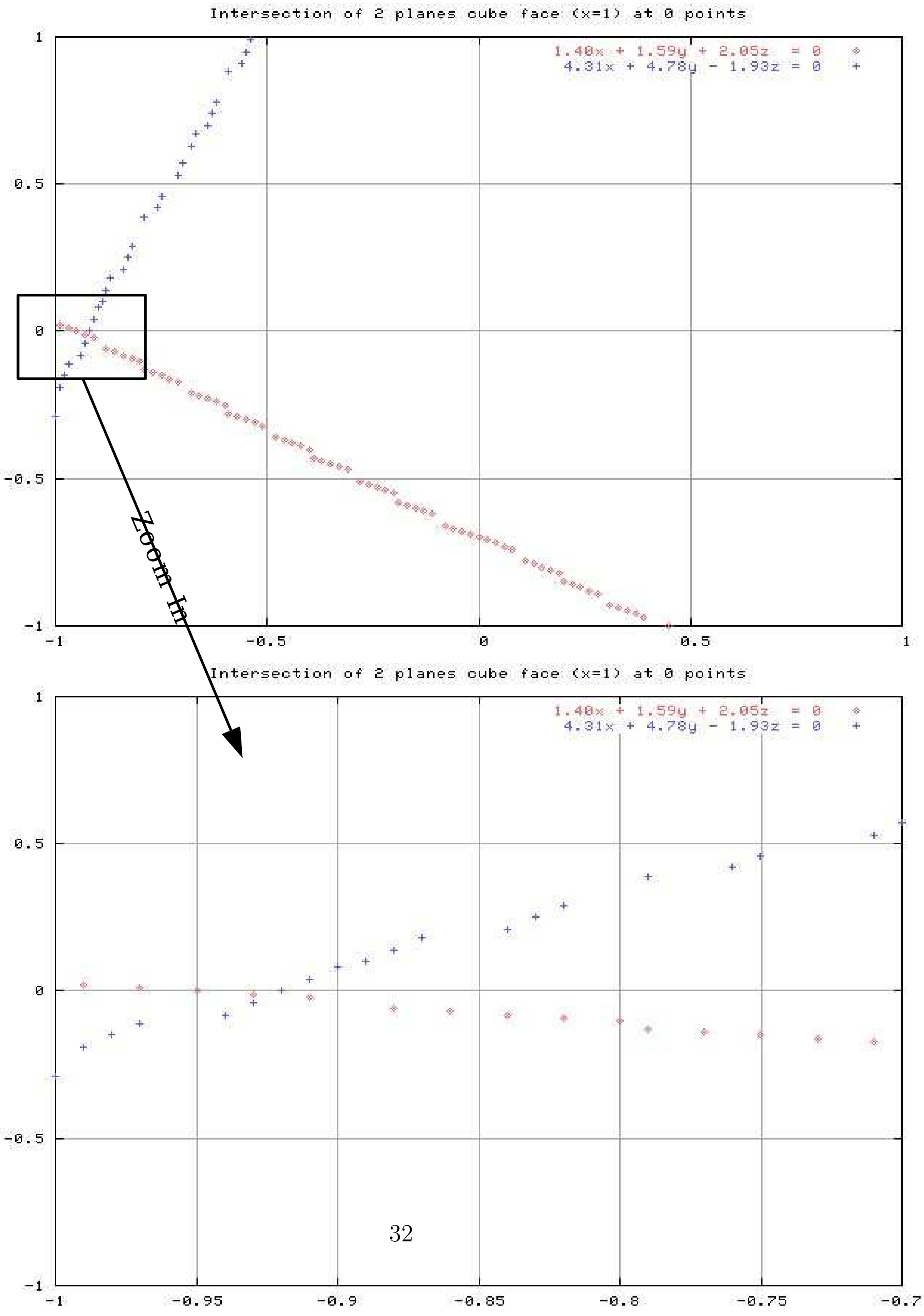
Intersection of 2 lines at 2 points



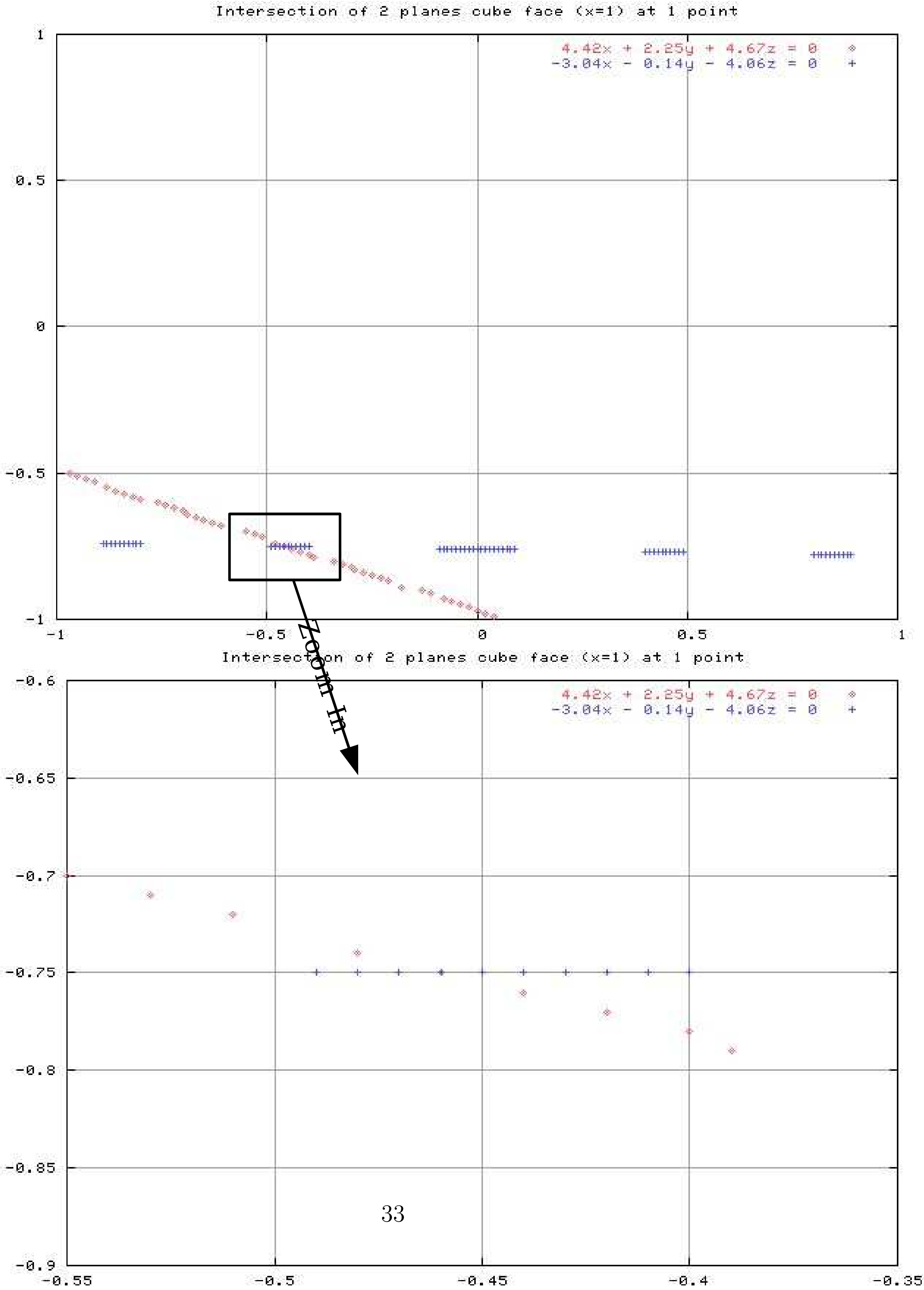
Intersection of 2 lines at 10 points



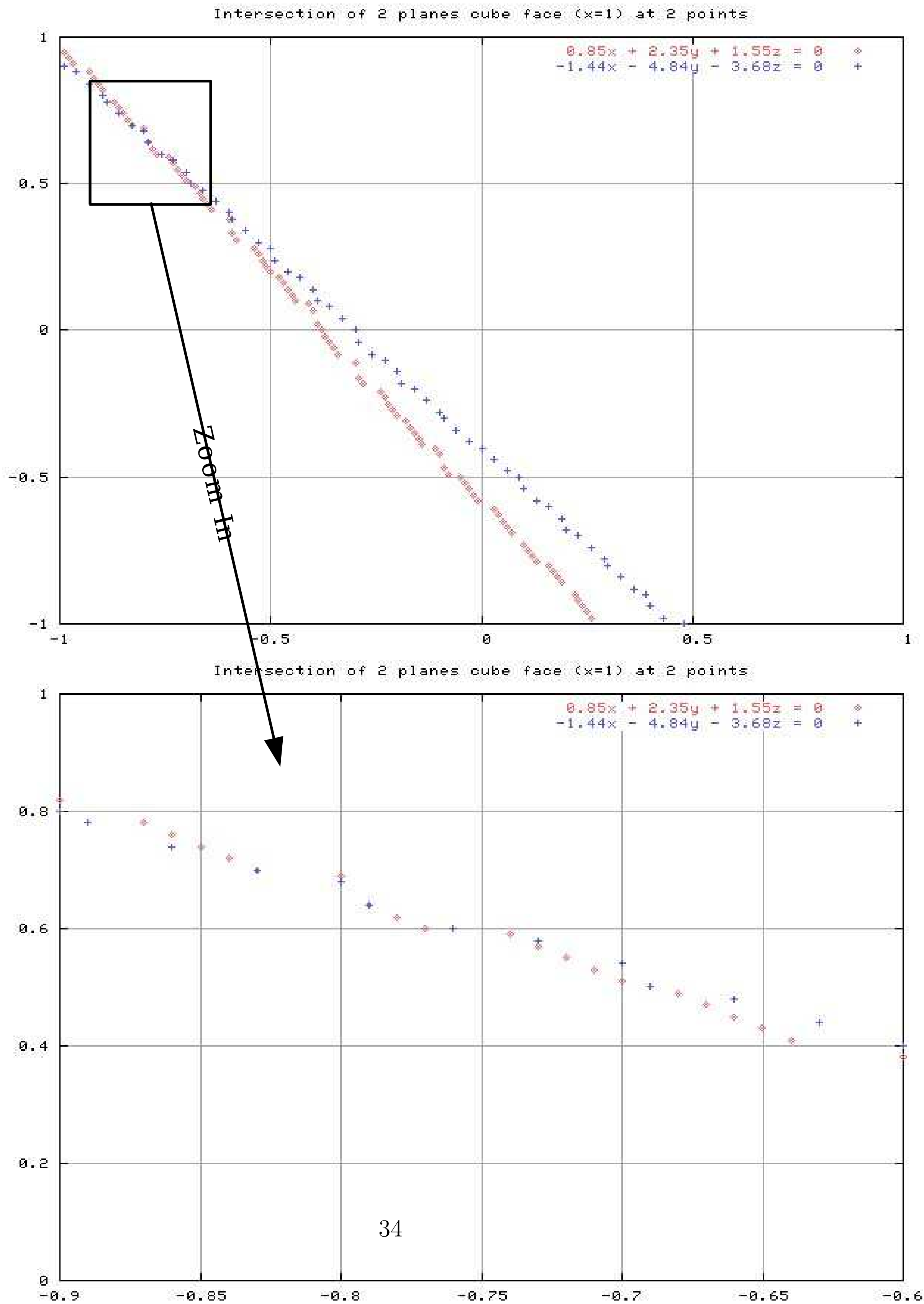
Intersection of 2 planes on cube face (x=1) at 0 points



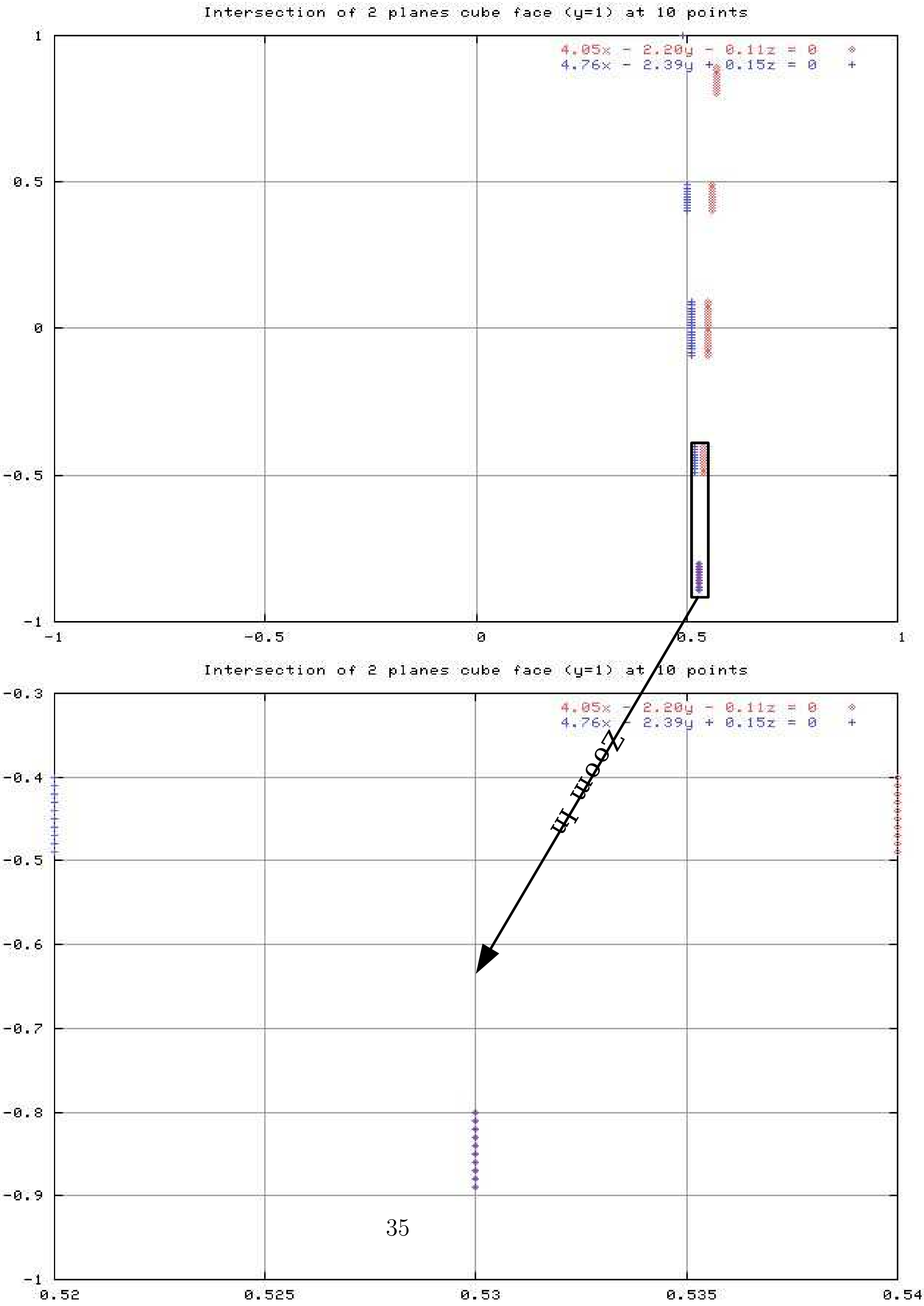
Intersection of 2 planes on cube face (x=1) at 1 point



Intersection of 2 planes on cube face (x=1) at 2 points



Intersection of 2 planes on cube face (y=1) at 10 points



Solving the system, we get $x = 0.5$ and $y = 1.5$, so the two lines intersect for any W_n. Now consider,

$$\begin{cases} y = 1.7 \times_2 x +_2 0.8 \\ y = 0.1 \times_2 x +_2 1.8 \end{cases}$$

To find this system's solution, consider its usual solution viewed in W_2, namely $(0.64, 1.86)$. Indeed, it is the solution of the given system. Now, when the solution is viewed in W_4, which is $(0.625, 1.8625)$, we also see that it is the solution, so the lines intersect for both $n = 2, 4$. For the next example, we consider the following two lines

$$\begin{cases} y = 31.85 \times_2 x +_2 1.28 \\ y = 7.41 \times_2 x +_2 7.12 \end{cases}$$

and consider their usual solution $(0.24, 8.88)$ By simply plugging in these values into the system, we see that they satisfy it. However, when the usual solution is viewed in W_4, we get $(0.2389, 8.8878)$ and it does not satisfy the system, however, even if we vary the $x-$coordinate's last decimal by unit increments, we find that there can be no solution for this system.

We now move to the discussion of Lobachevsky's and Riemannian geometries, see 6 and 7. Fix the$x-$axis, l_0, and pick a point on the $y-$axis, say $(0, b)$. Then we have the following theorem: the line $y = kx + b$ is parallel to l_0 in Lobachevsky sense iff $|b| \geq 1$, and in case $|b| < 1$, we would only have parallel lines in Euclidean sense.

In the new geometry however, there are many lines (not just in Lobachevsky or Euclidean sense) which do not intersect l_0 but go through the point$(0, b)$. For example, consider

$$\begin{cases} y = 3 \times_n x +_n b \\ y = 0 \end{cases}$$

then $3 \times_n x = -b$ and then, for instance, for $b = 1$, the system will not have a solution for any n. Obviously, for $b \neq 0$ we can always find (more than one) k, such that

$$\begin{cases} y = k \times_n x +_n b \\ y = 0 \end{cases}$$

does not have no solution. Consider W_2, then, in particular, for $b = 1, 1.01, ..., 1.98$, we find that $k = 0.01$. In fact, we have the following table for parallel lines:

b	k	Notes
0, 0.01,. . . , 0.99	0	Then there exits a unique line going through $(0, b)$ parallel to l_0, which is, in fact, parallel in Euclidean sense.
1, 1.01, . . . , 1.98	0.01	The lines going through $(0, b)$ are parallel to l_0 in Lobachevsky sense, and the lines for various values of b are parallel to each other in Euclidean sense.
1.99, 1.00, . . . , 2.97	0.02	...
2.98, 2.99,. . . , 3.96	0.03	...
. . .	. . .	...

This table was compiled using the fact that not intersecting the $x-$axis implies that the line gets as close as possible tol_0 in W_2, i.e. contains the point $(-99.99, 0.01)$ or $(-99.99, 0.02)$, etc.

Next, we consider Riemannian geometry, not as it is usually constructed, but as follows. Let us consider everything now with respect to W_2 and consider a unit cube centered at zero. Then the Riemannian line is an intersection of a plane containing the origin with faces of the cube. In the usual Riemannian geometry, any two such lines would intersect in two points, in the new geometry, however, the intersections may contain no, two, four, and twenty points, see figures above.

The classical Riemannian geometry (a line is an intersection of a plane containing the origin with the unit sphere) also changes in the new light. Consider two planes

$$\begin{cases} 0.2 \times_2 x -_2 0.1 \times_2 y -_2 10 \times_2 z = 0 \\ 0.3 \times_2 x -_2 0.2 \times_2 y -_2 36 \times_2 z = 0 \end{cases}$$

and the sphere $x^2 +_2 y^2 +_2 z^2 = 1$. These three sets intersect in 100 points. Here is an example of two planes intersecting the sphere in exactly two points

$$\begin{cases} x = 0 \\ y = 0 \\ x^2 +_2 y^2 +_2 z^2 = 0 \end{cases}$$

which are $(0, 0, \pm 1)$. The next system,

$$\begin{cases} x +_2 y -_2 4 \times_2 z = 0 \\ y -_2 4 \times_2 z = 0 \\ x^2 +_2 y^2 +_2 z^2 = 1 \end{cases}$$

implies that

$$\begin{cases} x = 0 \\ y = 4 \times_2 z \\ 17 \times_2 z^2 = 1 \end{cases}$$

and hence no solution, since 17^{-1} does not exist (see previous sections). Hence there are two lines that do not intersect in the new Riemannian sense.

2.5 Analysis and Topology

We begin the discussion of analysis with convergent sequences. Here is the definition of a sequence converging in some W_n from a point of view of W_m-observer with $m > n$. Given $x^k = x_0^k.x_1^k...x_n^k$ and $a = a_0.a_1...a_n$, with $k \in Z^+$, $x^k \to a$ (as k increases) iff there exist $k_0 < k_1 < ... < k_n \in W_n$ (integers) such that $x_0^k = a_0$ for all $k \geq k_0$, $\ldots$, $x_n^k = a_n$ for all $k \geq k_n$. Of course, locally, or for the W_n-observer, $x^k \to a$ convergence can be defined in the usual sense (via ε and N). Similarly, given $b = b_0.b_1...b_n$ and $f\left(x^k\right) = f_0^k.f_1^k...f_n^k$, we have $\begin{cases} f\left(x^k\right) \to b \\ x^k \to a \end{cases}$ iff there exist $k_0 < k_1 < ... < k_n \in W_n$ (integers), such that $\begin{cases} x_0^k = a_0 \\ f_0^k = b_0 \end{cases}$ for all $k \geq k_0$, $\ldots,$ $\begin{cases} x_n^k = a_n \\ f_n^k = b_n \end{cases}$ for all $k \geq k_n$. Again, for the W_n-observer the usual definitions take place. Now, from the point of view of the W_l-observer (with $l > m$), in W_m we have $x^k = x_0^k.x_1^k...x_m^k$, $a = a_0.a_1...a_m$, $b = b_0.b_1...b_m$ and $f\left(x^k\right) = f_0^k.f_1^k...f_m^k$ and therefore $\begin{cases} f\left(x^k\right) \to b \\ x^k \to a \end{cases}$ iff there exist $k_0 < k_1 < ... < k_m \in W_m-$integers), such that $\begin{cases} x_0^k = a_0 \\ f_0^k = b_0 \end{cases}$ for all $k \geq k_0$, $\ldots,$ $\begin{cases} x_m^k = a_m \\ f_m^k = b_m \end{cases}$ for all $k \geq k_m$. So, the W_m-observer sees infinity, while knows that the $W_n$observer does not. Similarly, the W_l-observer sees infinity, while knows that the other two do not. Let us consider how the W_l-observer studies the convergence: does $x \to a$ imply that$f(x) \to b$? The W_l-observer sees what happens with convergences in W_n and W_m, i.e. he pushes x towards a and studies what happens with $f(x)$ (whether it approaches b or not) in the $\varepsilon - \delta$ sense and also observes how this is seen by the W_n- and W_m-observers. Clearly, if W_l-observer sees that

$f(x) \to b$ in the $\varepsilon - \delta$ sense as $x \to a$, then there exist $k_0 < ... < k_n < ... < k_m < ... < k_l \in W_l$ such that $\begin{cases} x_0^k = a_0 \\ f_0^k = b_0 \end{cases}$ for all $k \geq k_0$, $\ldots$, $\begin{cases} x_n^k = a_n \\ f_n^k = b_n \end{cases}$ for all $k \geq k_n$,$\ldots$,$\begin{cases} x_m^k = a_m \\ f_m^k = b_m \end{cases}$ for all $k \geq k_m$, $\ldots$, $\begin{cases} x_l^k = a_l \\ f_l^k = b_l \end{cases}$ for all $k \geq k_l$ since the W_l-observer can take ε and δ such that $\varepsilon < 0.\underbrace{0...0}_{l}1$ and $\delta < 0.\underbrace{0...0}_{l}1$. However, it is not guaranteed that $k_0, k_1, ..., k_n \in W_n$ and $k_n, k_{n+1}, ..., k_m \in W_m$. This means that if $\begin{cases} f(x^k) \to b \\ x^k \to a \end{cases}$ for the W_l-observer, then it might not be true for the other observers. Also, if $\begin{cases} f(x^k) \to b \\ x^k \to a \end{cases}$ for the W_l-observer in the $\varepsilon - \delta$ sense we can have the following cases occur: $\begin{cases} f(x^k) \to b \\ x^k \to a \end{cases}$ for the other two observers, $\begin{cases} f(x^k) \to b \\ x^k \to a \end{cases}$ for W_n and $\begin{cases} f(x^k) \to b \\ x^k \to a \end{cases}$ for W_m-observers, and finally, $\begin{cases} f(x^k) \to b \\ x^k \to a \end{cases}$ for W_n and $\begin{cases} f(x^k) \to b \\ x^k \to a \end{cases}$ for W_m- observers (the divergence results are independent). Hence, the existence of a limit depends on the observer.

Let us now consider the following set of observers $W_{n_1}, \ldots, W_{n_{k+1}}$. Now, if we pick a $W_{n_{s+1}}$-observer with s large enough, such that he sees the following sequence: $a^r = \sum_{l=0}^{r} 10^{-2n_{k+1}+l}$ for $r = 0, 1, ..., 2n_{k+1}$ and $a^{2n_{k+1}+r} = a^{2n_{k+1}} + \sum_{l=1}^{r} 10^{-2n_{k+1}-l}$ for $r = 1, 2, ...$ (in particular, that means that this observer sees not only the entire $W_{n_{k+1}}$, but also the numbers $10^{-2n_{k+1}+l}$; also, we are using this notation to minimize space). Then this sequence converges to 1.1...1...1, i.e. for every

$W_{n_i}$the sequence stabilizes, but much later than it is possible to view even by the $W_{n_{s+1}}$ observer. This means that only some "future generation" of the $W_{n_{s+1}}$ observer will be able to see the complete stabilization of this sequence. The partial stabilization of the sequence, however, is seen by every generation of the $W_{n_{s+1}}$ observer, who thinks that this is the limit, which is not so.

In fact, this kind of convergence gives birth to the concept of time. Suppose we have $x, a, f(x), b \in W_{n_1}$ such that there exist $k_0, k_1, ..., k_{n_1}$ such that $\begin{cases} x_0^k = a_0 \\ f_0^k = b_0 \end{cases}$ for all $k \geq k_0$, $\ldots$, $\begin{cases} x_{n_1}^k = a_{n_1} \\ f_{n_1}^k = b_{n_1} \end{cases}$ for all $k \geq k_{n_1}$, but $k_0, k_1, ..., k_{n_1} \notin W_{n_1}$ or even $k_0, k_1, ..., k_{n_1} \notin W_{n_2}$, then, philosophically speaking, the W_{n_1}-, and W_{n_2}-observers do not see the fact of stabilization (convergence), but notice it only after some "time". Hence we call this phenomenon Time. Also, what is invalid for one generation can become valid for a union of a few generations (note here, we talk about a collection of these generations, not some single future generation). In general, an algorithm which contains not more than n steps is valid for only a single generation.

From the point of view of W_n-observer (we will call such observers "naive", since they "think" that they "live" in W and deal with W) a real function y of a real variable x, $y = y(x)$, is called differentiable at $x = x_0$ if there is a derivative

$$y'(x_0) = \lim_{x \to x_0, x \neq x_0} \frac{y(x) - y(x_0)}{x - x_0}$$

What does the above statement mean from point of view of W_m-observer with $m > n$? It means that

$$|(y(x) -_n y(x_0)) -_n (y'(x_0) \times_n (x -_n x_0))| \leq 0.\underbrace{0\ldots01}_{n}$$

whenever

$$|y(x) -_n y(x_0)| = 0.\underbrace{0\ldots0y_l}_{l}y_{l+1}\ldots y_n$$

and

$$|(x -_n x_0)| = 0.\underbrace{0\ldots0x_k}_{k}x_{k+1}\ldots x_n$$

for $1 \leq k$, $l \leq n$, and x_k - non-zero digit.

We now state the main theorems (see 8, 9, and 10).

THEOREM 2.7. *From the point of view of a W_m-observer a derivative calculated by a W_n-observer ($m > n$) is not defined uniquely.*

Proof. Put $y'(x_0) = \pm a_0.a_1\ldots a_p a_{p+1}\ldots a_n$ with $a_0.a_1\ldots a_p a_{p+1}\ldots a_n \geq 0$ and $p \leq n$. Then $0.\underbrace{0\ldots0y_l}_{l}y_{l+1}\ldots y_n = a_0.a_1\ldots a_p a_{p+1}\ldots a_n \times_n 0.\underbrace{0\ldots0x_k}_{k}x_{k+1}\ldots x_n = a_0.a_1\ldots a_p b_{p+1}\ldots b_n \times_n 0.\underbrace{0\ldots0x_k}_{k}x_{k+1}\ldots x_n$ for any digits $b_{p+1},\ldots,b_n$ and $p = n-k$. Hence

$$y'(x_0) \in V = \{\pm a_0.a_1\ldots a_p a_{p+1}\ldots a_n | a_{p+1},\ldots,a_n \in \{0,1,\ldots,9\}\}$$

and $|V| = 10^k$. □

THEOREM 2.8. *From the point of view of a W_m-observer with $m > n$, $|y'(x_0)| \leq C_n^{l,k}$, where $C_n^{l,k} \in W_n$ is a constant defined only by n, l, k and not dependent on $y(x)$.*

Proof. We have $\pm 0.\underbrace{0\ldots0y_l}_{l}y_{l+1}\ldots y_n = (\pm a_0.a_1\ldots a_n) \times_n (\pm 0.\underbrace{0\ldots0x_k}_{k}x_{k+1}\ldots x_n)$ with x_k - non-zero digit and $a_0.a_1\ldots a_p a_{p+1}\ldots a_n \geq 0$. Now, if $l > k$ then $a_0 = 0$; if $l = k$ then $a_0 \leq 9$

and if $l < k$ then $a_0 < 9 \times 10^{k-1}$. Hence

$$C_n^{l,k} = \begin{cases} 1, \text{ if } l > k \\ 10, \text{ if } l = k \\ 9 \times 10^{k-1}, \text{ if } l < k \end{cases}$$

∎

THEOREM 2.9. *From the point of view of a W_m-observer, when a W_n-observer (with $m > n \geq 3$) calculates the second derivative:*

$$y''(x_0) = \lim_{x_1 \to x_0, x_1 \neq x_0, x_2 \to x_0, x_2 \neq x_0, x_3 \to x_1, x_3 \neq x_1} \frac{\frac{y(x_3)-y(x_1)}{(x_3-x_1)} - \frac{y(x_2)-y(x_0)}{x_2-x_0}}{x_1 - x_0}$$

we get the following unequality:

$$(|x_2 -_n x_0| \times_n |x_3 -_n x_1|) \times_n |x_1 -_n x_0| \geq 0.\underbrace{0\ldots01}_{n}$$

provided that $y''(x_0) \neq 0$.

Proof. For the W_m-observer existence of $y''(x_0)$ means that $|((y(x_3) -_n y(x_1)) \times_n (x_2 -_n x_0) -_n ((y(x_2) -_n y(x_0)) \times_n (x_2 -_n x_0))) -_n y''(x_0) \times_n ((|x_2 -_n x_0| \times_n |x_3 -_n x_1|) \times_n |x_1 -_n x_0|)| \leq 0.\underbrace{0\ldots01}_{n}$, whenever

$$|(x_2 -_n x_0)| \leq 0.\overbrace{\underbrace{0\ldots0}_{k}p*\ldots*}^{n}$$

and

$$|(x_3 -_n x_1)| \leq 0.\overbrace{\underbrace{0\ldots0}_{l}q*\ldots*}^{n}$$

and

$$|(x_1 -_n x_0)| \leq 0.\overbrace{\underbrace{0\ldots0}_{s}r*\ldots*}^{n}$$

where p, q, r are non-zero digits, asterisks are any digits and $3 \leq k + l + s \leq n$. Then given $y''(x_0) \neq 0$ we have $(|x_2 -_n x_0| \times_n |x_3 -_n x_1|) \times_n |x_1 -_n x_0| \geq 0.\underbrace{0\ldots01}_{n}$. □

The following hypotheses illustrate possible physical interpretation of previous theorems.

Hypothesis 1. Theorem 2.7 could offer an explanation of why physical speed (or acceleration) is not uniquely defined and, from the point of view of a measurement system (observer), it is possible to consider speed (or acceleration) as a random variable with distribution dependend on the measurement system. Let v be the speed with $v = v_0.v_1 \ldots v_{n-k} + \xi_m^{n,k}$ where $\xi_m^{n,k} \in \{0.\underbrace{0\ldots0}_{n-k} v_{n-k+1} \ldots v_n\}$ - random variable, $m > n$, and the distribution function is $F_m^{n,k}(x) = P(\xi_m^{n,k} < x)$.

Hypothesis 2. Theorem 2.8 could offer an explanation of why the speed of any physical body cannot exceed some constant, (the speed of light, for example). Independence of this constant on explicit expression of space-time function could offer an explanation of why the speed of light does not depend on an inertial coordinate system.

Hypothesis 3. Theorem 2.9 could offer an explanation of the various uncertainty principles, when a product of a finite number of physical variables has to be not less than a certain constant. This can be seen not just from consideration of second derivatives, but of any derivative.

The Cauchy-Kowalevski theorem is the main local existence and uniqueness theorem for analytic partial differential equations associated with Cauchy initial value problems. A special case was proved by Augustin Cauchy, see 11, and the full result by Sophie Kowalevski, see 12. The first order Cauchy-Kowalevski theorem is about the existence of solutions to a system of m differential equations in n dimensions when the coefficients are analytic functions. The theorem and its proof are valid for analytic functions of either real or complex variables.

Let K denote either the fields of real or complex numbers and let $V = K^m$ and $W = K^n$. Let $A_1, \ldots, A_{n-1}$ be analytic functions defined on some neighborhood of $(0,0)$ in $V \times W$ and taking values in the $m \times m$ matrices, and let b be an analytic function with values in V on the same neighborhood. Then there is a neighborhood of 0 in W on which the quasilinear Cauchy problem

$$\partial_{x_n} f = A_1(x,f)\partial_{x_1} f + \ldots + A_{n-1}(x,f)\partial_{x_{n-1}} f + b(x,f)$$

with initial condition $f(x) = 0$ on the hypersurface $x_n = 0$ has a unique analytic solution $f : V \to W$ near 0.

Lewy's example shows that the theorem is not valid for all smooth functions. The theorem can also be stated in abstract (real or complex) vector spaces. Let V and W be finite-dimensional real or complex vector spaces, with $n = dimW$. Let $A_1, \ldots, A_{n-1}$ be analytic functions with values in $End(V)$ and b an analytic function with values in V, defined on some neighborhood of $(0,0)$ in $V \times W$. In this case, the same result holds.

The higher-order Cauchy-Kowalevski theorem can be stated as follows. If F and f_j are analytic functions near 0, then the non-linear Cauchy problem $\partial_t^k h = F(x, t, \partial_t^j, \partial_x^\alpha h)$, where $j < k$ and $|\alpha| + j \leq k$, with initial conditions $\partial_t^j h(x,0) = f_j(x)$, with $0 \leq j < k$, has a unique analytic solution near 0. This follows from the first order problem by considering the derivatives of h appearing on the right hand side as components of a vector-valued function.

Considered in the following sections analysis of concepts such as Free Wave equation, Schrodinger equation, two-slit interference, wave-particle duality for single photons, uncertainty principle, Airy and Korteweg-de Vries equations, and Dirac equation, Newton equation, shows that in Observers's Mathematics Cauchy-Kowalevski theorems become invalid, see [12]. Instead, we have stochastic properties of partial (and ordinary) differential equations, both linear and non-linear.

2.6 Logic

In the new setting of different observers some facts in usual logic can become invalid, see 13 and 14. For example, the axiom of choice is not valid. Consider W_2, this set has 10^4 non-negative elements, however, the largest number that a W_2-observer can see is 99 (from a point of view of an observer with a larger thickness number), therefore, picking any element from W_2 will take more steps than what is allotted for the W_2-observer, hence impossible. However, there is an interesting chain of the sets W, namely $W_2 \subset W_4 \subset W_8 \subset ... \subset W_{2^n} \subset ....$ For this chain, it is clear that the Axiom of Choice is valid for when picking an element from $W_{2^{n-1}}$by a W_{2^n}(or higher) observer. Moreover, we can always pick *any* element from W_n, whenever we are in W_{2n}.

In 2.5, we introduced the concept of time. Then we can see that the Axiom of Choice is would be valid in the following sense: an observer will see any element after $\frac{2^n}{n}$ time increments, but this observer will not know about it, only the observer that sees the number 10^{2^n}.

Next, we have the paradox of the set of all sets (consider the set of all subsets of that set). In the new light, there is no such paradox, due to the fact that a given observer will not be able to see all the subsets, i.e. a W_n-observer can only see 10^n elements, which have 2^{10^n} subsets, so only an observer with larger thickness values would be able to see it.

3. NUMBER THEORY FROM OBSERVER'S MATHEMATICS POINT OF VIEW

First we will prove the following four Theorems:

Theorem 3.1. *(Analogy of Fermat's Last Problem). For any integer n, $n \geq 2$, and for any integer m, $m \geq 3$, $m \in W_n$ (see below for the definition of W_n) there exist positive a, b, $c \in W_n$, such that $a^m +_n b^m = c^m$ (operation $+_n$ is defined below).*

Theorem 3.2. *(Analogy of Mersenne's numbers problem). There exist integers n, $k \geq 2$, Mersenne's numbers M_k, with $\{k, M_k\} \in W_n$, and positive $a \in W_n$, such that $M_k = a^2$.*

Theorem 3.3. *(Analogy of Fermat's numbers problem). There exist integers n, $k \geq 2$, Fermat's numbers F_k, $\{k, F_k\} \in W_n$, and positive $a \in W_n$, such that $F_k = a^2$.*

Theorem 3.4. *(Analogy of Waring's problem). For any integer k, $k \geq 2$, there exist integer n, $n \geq 2$, $(k \in W_n)$ and some $x \in W_n$ such that any equality of the form $x = a_1^k + a_2^k + \ldots + a_l^k$ is not possible for any integer $l \in W_n$ and any positive numbers a_1, a_2, $\ldots$, $a_l \in W_n$.*

3.1 Analogy of Fermat's Last Problem

To begin, we present a few notes, see 13 and 15. It is obvious that the classical Fermat's Last Problem (for any integer m, $m \geq 3$, there do not exist positive integers a, b, c, such that $a^m + b^m = c^m$) may be reformulated not just for integers a, b, c, but for any real rational numbers a, b, c.

Note, in Observer's Mathematics the power operation is not always associative. For illustrative purposes, we give a W_2 example. Consider $1.49 \in W_2$. Then $1.49 \times_2 1.49 = 2.14$ and $1.49 \times_2 2.14 = 3.16$. On the other hand, $1.49 \times_2 3.16 = 4.67$ and $2.14 \times_2 2.14 = 4.57$, i.e. $((1.49 \times_2 1.49) \times_2 1.49) \times_2 1.49 \neq (1.49 \times_2 1.49) \times_2 (1.49 \times_2 1.49)$.

We now re-state Theorem 3.1:

For any integer n, $n \geq 2$, and for any integer m, $m \geq 3$, $m \in W_n$ there exist positive a, b, $c \in W_n$, such that $a^m +_n b^m = c^m$. Here x^m means $\underbrace{((\ldots(x \times_n x) \times_n \ldots) \times_n x))}_{m}$

Proof. Put $a = b = c = 0.\underbrace{0\ldots0}_{k}1$ where $1 \leq k \leq m$, $k \times m \in W_n$, $k \times m > n$ (note, "$\times$" is the multiplication sign in standard arithmetic). Then $a^m = b^m = c^m = 0$, hence $a^m +_n b^m = c^m$. □

Note that the full set of solutions of the equation $a^m +_n b^m = c^m$ is not a simple problem.

For example, if $n = 2$, we can calculate that

$1^3 +_2 1^3 = 1.28^3$

$1^3 +_2 1.21^3 = 1.41^3$

$1.2^3 +_2 1.03^3 = 1.41^3$

$1^{20} +_2 1^{20} = 1.05^{20}$

$1^{25} +_2 1^{25} = 1.04^{25}$

$1^{50} +_2 1^{50} = 1.02^{50}$.

For $n = 3$, we can calculate that

$1^{17} +_3 1^{17} = 1.044^{17}$

$1^{22} +_3 1^{22} = 1.034^{22}$

$1^{50} +_3 1^{50} = 1.016^{50}$

$1^{200} +_3 1^{200} = 1.005^{200}$

$1^{250} +_3 1^{250} = 1.004^{250}$

$1^{500} +_3 1^{500} = 1.002^{500}$.

For $n = 4$, we can calculate that

$1^{2000} +_4 1^{2000} = 1.0005^{2000}$

$1^{2500} +_4 1^{2500} = 1.0004^{2500}$

$1^{5000} +_4 1^{5000} = 1.0002^{5000}$

For $n = 8$, we can calculate that

$1.8601023^3 +_8 1.35432561^3 = 2.07390372^3$

$1.02345678^3 +_8 1.25160402^3 = 1.44746886^3$

$1.13687002^3 +_8 1.57041392^3 = 1.74814264^3$

$1.00056781^4 +_8 1.42300976^4 = 1.50297066^4$

$1.85643209^4 +_8 1.67843218^4 = 2.10979538^4$

$1.85643209^5 +_8 1.55566643^5 = 1.98939654^5$.

For $n = 16$, we can calculate that

$1.4230990164830891^3 +_{16} 1.5704139255639073^3 = 1.8903509118894252^3$

Note that the main reason of cardinal difference between standard Mathematics and Observer's Mathematics results is the following. The negative solution of classical Fermat's problem requires Axiom of Choice to be valid. But in Observer's Mathematics this Axiom is invalid.

3.2 Analogy of Mersenne's and Fermat's Numbers Problems

Mersenne's numbers are defined as $M_k = 2^k - 1$, with $k = 1, 2, \ldots$. The following question is still open: is every Mersenne's number square-free?

Fermat's numbers are defined as $F_k = 2^{2^k} + 1$, $k = 0, 1, 2, \ldots$. The following question is still open: is every Fermat's number square-free?

We begin with some comments, see 15. It is obvious that if some integer number is square-free in the set of all real integers, than this number is square-free in the set of all real rational numbers. We note re-state Theorem 3.2:

There exist integers n, $k \geq 2$, Mersenne's numbers M_k, with $\{k, M_k\} \in W_n$, and positive $a \in W_n$, such that $M_k = a^2$.

Proof. For $k = 3$, $M_3 = 7$. Then for $n = 2$ the set $\{3, 7\} \subset W_2$ and $2.66 \times_2 2.66 = 7$, i.e., $a = 2.66$. For $k = 2$, $M_2 = 3$. Then for $n = 3$ the set $\{2, 3\} \subset W_3$ and $1.734 \times_3 1.734 = 3$, i.e., $a = 1.734$. For $k = 3$, $M_3 = 7$. Then for $n = 3$ the set $\{3, 7\} \subset W_3$ and $2.648 \times_3 2.648 = 7$, i.e., $a = 2.648$. For $k = 5$, $M_5 = 31$. Then for $n = 3$ the set $\{5, 31\} \subset W_3$ and $5.569 \times_3 5.569 = 31$, i.e., $a = 5.569$. For $k = 8$, $M_8 = 255$. Then for $n = 4$ the set $\{8, 255\} \subset W_4$ and $15.9688 \times_4 15.9688 = 255$, i.e., $a = 15.9688$. $\square$

We note re-state Theorem 3.3, see 15.

There exist integers n, $k \geq 2$, Fermat's numbers F_k, $\{k, F_k\} \in W_n$, and positive $a \in W_n$, such that $F_k = a^2$.

Proof. For $k = 1$, $F_1 = 5$. Then for $n = 2$ the set $\{1, 5\} \subset W_2$ and $2.24 \times_2 2.24 = 5$, i.e., $a = 2.24$. For $k = 0$, $F_1 = 3$. Then for $n = 3$ the set $\{0, 3\} \subset W_3$ and $1.734 \times_2 1.734 = 3$, i.e.,

$a = 1.734$. For $k = 1$, $F_1 = 5$. Then for $n = 3$ the set $\{1, 5\} \subset W_3$ and $2.237 \times_2 2.237 = 5$, i.e., $a = 2.237$. For $k = 0$, $F_1 = 3$. Then for $n = 5$ the set $\{0, 3\} \subset W_5$ and $1.73209 \times_2 1.73209 = 3$, i.e., $a = 1.73209$. For $k = 3$, $F_1 = 257$. Then for $n = 5$ the set $\{3, 257\} \subset W_5$ and $16.03122 \times_2 16.03122 = 257$, i.e., $a = 16.03122$. □

3.3 Analogy of Waring's Problem

It is known (Lagrange) that the minimum number of squares to express all positive integers is four. What is the minimum number of k-th powers necessary to express all positive integers? This is a classical Waring's problem, in standard arithmetic.

We now re-state Theorem 3.4:

For any integer k, $k \geq 2$, there exist integer n, $n \geq 2$, ($k \in W_n$) and some $x \in W_n$ such that any equality of the form $x = a_1^k + a_2^k + \ldots + a_l^k$ is not possible for any integer $l \in W_n$ and any positive numbers $a_1, a_2, \ldots, a_l \in W_n$.

Proof. For given integer $k \geq 2$ take $n = k + 1$. So, $k \in W_n$, and take $x = 0.\underbrace{0\ldots0}_{k}1$. There is no positive $a \in W_n$ such that $a^k = x$, because a^k equals to 0 or is greater than x. Hence, if we have any integer $l \geq 1$ and any positive $a_1, a_2, \ldots, a_l \in W_n$, we see that $a_1^k + a_2^k + \ldots + a_l^k$ equals to 0 or is greater than x. $\square$

Note that for $n = 2$ and for any $x \in W_2$, $x \in [0, 1]$, there do not exist more than four numbers $a, b, c, d \in W_2$, such that $x = ((a^2 +_2 b^2) +_2 c^2) +_2 d^2$. It is easy to check this statement by direct calculation.

3.4 Analogy of Hilbert's Tenth Problem

Consider the equation:

$$x^3 + y^3 + z^3 = 29$$

Does this equation have an integer solution? The answer is yes, it does: for example, $\{3, 1, 1\}$. What about the equation:

$$x^3 + y^3 + z^3 = 30$$

The answer is again yes. The solution $\{-283059965, -2218888517, -2220422932\}$ was found only in 1999. What about the equation:

$$x^3 + y^3 + z^3 = 33$$

This is an unsolved problem. D. Hilbert had asked in 1900 to find a method that would answer all such questions. Hilbert's tenth problem asks for an algorithm that takes as input a multivariable polynomial $f(x_1, \ldots, x_k)$ with integer coefficients and output YES or NO according to whether there exist integers $a_1, \ldots, a_k$ such that $f(a_1, \ldots, a_k) = 0$. In 1970, Yu. Matiyasevich, building on earlier work of M. Davis, H. Putnam and J. Robinson, showed that no such algorithm exists.

We consider here Hilbert's tenth problem from the point of view of Observer's Mathematics, see 14 and 16.

We are looking for the solution of equation

$$x^3 + y^3 + z^3 = 33$$

in W_n, $n \geq 2$, i.e. we have to find $x, y, z \in W_n$ such that $((x \times_n x) \times_n x)$, $((y \times_n y) \times_n y)$, $((z \times_n z) \times_n z)$, $((x \times_n x) \times_n x) +_n ((y \times_n y) \times_n y) \in Wn$, and $(((x \times_n x) \times_n x) +_n ((y \times_n y) \times_n y)) +_n ((z \times_n z) \times_n z) = 33$. We provide several solutions below:

1. For $n = 2$, the solutions are:

   (a) $\{1.72, 1, 3\}$,

   (b) $\{-1.28, 2, 3\}$,

   (c) $\{2.37, 1.55, 2.54\}$.

2. For $n = 3$, the solutions are:

   (a) $\{3.208, 0, 0\}$,

   (b) $\{3.208, y, -y\}$ for any $y \in W_3$ such that $(y \times_3 y) \times_3 y$, $(y \times_3 y) \times_3 y + 33 \in W_3$,

   (c) $\{2.887, 1, 2\}$.

3. For $n = 4$, a possible solution is: $\{2.4102, -2, 3\}$.

4. For $n = 5$, a possible solution is: $\{4.12129, 3, -4\}$.

5. For $n = 6$, the solutions are:

   (a) $\{2.8845, 1, 2\}$,

   (b) $\{1.709981, 1, 3\}$.

6. For $n = 9$, a possible solution is: $\{2.571281595, -1, -1\}$.

7. For $n = 10$, a possible solution is: $\{3.8929964162, 1, -3\}$.

8. For $n = 11$, a possible solution is: $\{3.89299641591, 1, -3\}$.

9. For $n = 12$, a possible solution is: $\{3.659305710025, -2, -2\}$.

10. For $n = 13$, the solutions are:

    (a) $\{2.9240177382132, 0, 2\}$,

    (b) $\{2.9240177382132, 2, 0\}$.

11. For $n = 14$, the solutions are:

    (a) $\{4.08165510191737, -2, -3\}$,

    (b) $\{4.71769398031657, -2, -4\}$.

12. For $n = 15$, the solutions are:

    (a) $\{2.410142264175234, -2, 3\}$,

    (b) $\{1.259921049894891, 2, 3\}$,

    (c) $\{4.081655101917351, -2, -3\}$.

In W_n, from the W_n-observer's point of view (W_n-observer is "naive" in W_n), Hilbert's Tenth Problem is formulated classically: "Is there an algorithm that takes as input a multivariable polynomial $f(x_1, \ldots, x_k)$ with integer coefficients and outputs YES or NO according to whether there exist integers $a_1, \ldots, a_k$ such that $f(a_1, \ldots, a_k) = 0$." And W_n-observer as "naive" one has and understands proof, which Yu. Matiyasevich based on works of M. Davis, H. Putham, and J. Robinson made in 1970, and shown that no such algorithm exists. Consider now what does it mean from W_m-observer's point of view ($m > n$).

First we address the question "what is a polynomial in W_n" from the point of view of W_m-observer, with $m > n$?

DEFINITION 3.5. *Multivariable (k - variables) polynomial $f(x_1, \dots, x_k)$ with degree q in W_n is given by:* $\sum_{p=0}^{q} {}^n \sum_{i_1+\dots+i_k=p} {}^n a_{i_1 \dots i_k} \times_n (\dots(\dots(\underbrace{x_1 \times_n x_1) \times_n \dots) \times_n x_1}_{i_1}) \times_n \dots$

$\times_n(\dots(\underbrace{x_k \times_n x_k) \times_n \dots) \times_n x_k}_{i_k})$ *where $k \in N \cap W_n$, $q, i_1, \dots, i_k \in (N \cup 0) \cap W_n$, with N - the set of all positive integers, and $a_{i_1 \dots i_k}, x_1, \dots, x_k$ and entries of all parentheses are in W_n.*

THEOREM 3.6. *For any positive integers $m, n, k \in W_n$, $n \in W_m$, $m > log_{10}(1 + (2 \cdot 10^{2n} - 1)^k)$, from the point of view of the W_m-observer, there is an algorithm that takes as input a multivariable polynomial $f(x_1, \dots, x_k)$ of degree q in W_n and outputs YES or NO according to whether there exist $a_1, \dots, a_k \in W_n$ such that $f(a_1, \dots, a_k) = 0$.*

Proof. From the point of view of a $W_m$observer, the set of all possible vectors $(a_1, \dots, a_k)$ is finite. The power of this set is $(2 \cdot 10^{2n} - 1)^k$. Since $m > log_{10}(1 + (2 \cdot 10^{2n} - 1)^k)$, a brute force algorithm would work. □

Note, that, for example, for $n = 2$ and $k = 3$, this problem has negative solution from the point of view of not only W_2-observer, but also for W_3,W_4,. . .,W_{12}-observers and only from the point of view of W_m-observer with $m \geq 13$ this problem has positive solution.

Therefore, Hilbert's tenth problem in Observer's Mathematics has positive solution. We think that Hilbert expected a positive answer for his tenth problem. Note, that the main reason of cardinal difference between standard Mathematics and Observer's Mathematics results is the following. The negative solution of classical tenth problem requires Axiom of Choice to be valid. But in Observer's Mathematics this Axiom is invalid.

3.5 Lehmer's Number in Observer's Mathematics

Lehmer's number, $\alpha \approx = 1.17628$, is the largest real root of the polynomial $f(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$. This number appears in various contexts in number theory and topology as the (sometimes conjectural) answer to natural questions involving notions of "minimality" and "small complexity".

A Salem polynomial (see 17) is a monic irreducible reciprocal polynomial $\phi(x)$ in $\mathbb{Z}[x]$ such that $\phi(x) = 0$ has exactly two real roots $\alpha > 1$ and $1/\alpha$ off the unit circle $S^1 := \{z \in \mathbb{C}||z|\}$. Is is then of even degree. A Salem number is the unique real root $\alpha > 1$. In other words, a Salem number of degree $2n$ is a real algebraic integer $\alpha > 1$ whose Galois conjugates consist of $1/\alpha$ and $2n-2$ imaginary numbers on S^1.

There are 47 known Salem numbers less than 1.3. Of these, 45 exhaust the possibilities with $\alpha < 1.3$ and degree $d < 42$. Of these, merely 6 have degree $d < 12$. Of these 6, we noted that all but one solve equation of the very simple form $x^{4+m} = \frac{Q(1/x)}{Q(x)}$ with $m > 0$ and $Q(x) = x^3 - x - 1$. The case $m = 1$ gives Lehmer's number field. The minimal polynomials of the first five Salem numbers in this family are

$$P_1(\alpha) = \alpha^{10} + \alpha^9 - \alpha^7 - \alpha^6 - \alpha^5 - \alpha^4 - \alpha^3 + \alpha + 1$$

$$P_2(\alpha) = \alpha^{10} - \alpha^7 - \alpha^5 - \alpha^3 + 1$$

$$P_3(\alpha) = \alpha^{10} - \alpha^8 - \alpha^5 - \alpha^2 + 1$$

$$P_4(\alpha) = \alpha^8 - \alpha^5 - \alpha^4 - \alpha^3 + 1$$

$$P_5(\alpha) = \alpha^{10} - \alpha^8 - \alpha^7 + \alpha^5 - \alpha^3 - \alpha^2 + 1$$

with approximate numerical roots - and hence Mahler measures - given by

$$\alpha_1 = 1.17628081825991750654407033 84\ldots$$

$$\alpha_2 = 1.23039143440722470279017793 89\ldots$$

$$\alpha_3 = 1.26123096113713885194667150 30\ldots$$

$$\alpha_4 = 1.28063815626775759670190253 27\ldots$$

$$\alpha_5 = 1.29348595312545410651990988 37\ldots$$

Prior to understanding the Lehmer-Salem numbers in the setting of Observer's Mathematics, we first address the question "what is a polynomial in W_n?".

DEFINITION 3.7. *Polynomial $f(x)$ with degree q in W_n is given by the following formula:*

$$f(x) = \sum_{p=0}^{q} {}^{n} a_p \times_n (\ldots (x \times_n x) \times_n x) \times_n \ldots \times_n x)$$

where q, a_p, x and entries of all parentheses are in W_n, $p = 0, 1, \ldots, q$.

Note that exponent is not an associative operation. For example, for $n = 2$, we have $1.49 \times_2 1.49 = 2.14$, $1.49 \times_2 2.14 = 3.16$, $1.49 \times_2 3.16 = 4.67$, i.e., $((1.49 \times_2 1.49) \times_2 1.49) \times_2 1.49 = 4.67$ while $(1.49 \times_2 1.49) \times_2 (1.49 \times_2 1.49) = 4.57$.

In W_n we can define a root of a polynomial $f(x) \in W_n$ as the number $x_0 \in W_n$ such that $|f(x)| \leq 0.\underbrace{0\ldots01}_{n}$.

THEOREM 3.8. *There are some $n \in \mathbb{N}$ such that the minimal polynomial of the first five Salem numbers have the roots in W_n. Note that we consider $P_1(\alpha), P_3(\alpha), P_4(\alpha)$ as the polynomial in*

W_n in the sense of the definition above, i.e., we understand that $x^k = (\ldots \underbrace{(x \times_n x) \times_n \ldots \times_n x)}_{k}$ *and* $\sum = \sum^n$.

Proof. Using direct calculations we can see the following.

For $P_1(x)$:

- $n = 7$, $x_0 = 1.1762812$
- $n = 9$, $x_0 = 1.176280822$

For $P_3(x)$:

- $n = 10$, $x_0 = 1.2612309614$

For $P_4(x)$:

- $n = 9$, $x_0 = 1.280638160$

□

THEOREM 3.9. *For $n \leq 10$ there do not exist roots for $P_2(x)$ and $P_5(x)$.*

Proof. Direct calculation. □

THEOREM 3.10. *Unramified minimal polynomial* $S(x) = 1 - x + x^2 - x^3 - x^6 + x^7 - x^8 + x^9 - x^{10} + x^{11} - x^{12} - x^{15} + x^{16} - x^{17} + x^{18}$ *which defined, after Lehmer's number the second smallest known Salem number* $\alpha \approx 1.188368$*, has a root in* W_8*:* $x_0 = 1.18836817$*.*

Proof. Direct calculation. □

3.6 Euler Brick and Perfect Cuboid problems

An Euler Brick is just a cuboid, or a rectangular box, in which all of the edges (length, depth, and height) have integer dimensions and in which the diagonals on all three sides are also integers. So if the length, depth and height are a, b, and c respectively, then a, b, and c are integers, as are the quantities $\sqrt{a^2+b^2}$, $\sqrt{b^2+c^2}$ and $\sqrt{c^2+a^2}$. The unsolved problem is to find a four dimensional Euler Brick, in which the four sides a, b, c, and d are integers, as are the six face diagonals $\sqrt{a^2+b^2}$, $\sqrt{a^2+c^2}$, $\sqrt{a^2+d^2}$, $\sqrt{b^2+c^2}$, $\sqrt{b^2+d^2}$ and $\sqrt{c^2+d^2}$, or prove that such a cuboid cannot exist.

We reformulate $4D$ Euler Brick problem for Observer's Mathematics in the following way. To find some positive integer n and a $4D$ cuboid, in which the four sides a, b, c, d are integers in W_n, and the six face diagonals $\sqrt{a^2+b^2}$, $\sqrt{a^2+c^2}$, $\sqrt{a^2+d^2}$, $\sqrt{b^2+c^2}$, $\sqrt{b^2+d^2}$ and $\sqrt{c^2+d^2}$ are also in W_n, or prove that such cuboid cannot exist.

THEOREM 3.11. *If $a=b=c=d=1$, then the following condition holds true in W_2*

$$\sqrt{a^2+b^2}=\sqrt{a^2+c^2}=\sqrt{a^2+d^2}=\sqrt{b^2+c^2}=\sqrt{b^2+d^2}=\sqrt{c^2+d^2}=1.42\in W_2$$

Also, the following condition holds true in W_3:

$$\sqrt{a^2+b^2}=\sqrt{a^2+c^2}=\sqrt{a^2+d^2}=\sqrt{b^2+c^2}=\sqrt{b^2+d^2}=\sqrt{c^2+d^2}=1.416\in W_3$$

The above theorem implies that this problem has a positive solution in Observer's Mathematics.

Another unsolved problem is to find a perfect cuboid, which is an Euler Brick in which the space diagonal, that is, the distance from any corner to its opposite corner, given by the formula $\sqrt{a^2+b^2+c^2}$, is also an integer, or prove that such a cuboid cannot exist.

We reformulate perfect cuboid problem for Observer's Mathematics in the following way. To find some positive integer n and a perfect cuboid, in which the three sides a, b, c are integers in W_n, and the three face diagonals $\sqrt{a^2+b^2}$, $\sqrt{a^2+c^2}$ and $\sqrt{b^2+c^2} \in W_n$, and in which the space diagonal, that is, the distance from any corner to its opposite corner, given by the formula $\sqrt{a^2+b^2+c^2} \in W_n$, or prove that such a cuboid cannot exist.

THEOREM 3.12. *If $a = b = c = 1$, then the following condition holds true in W_2:*

$$\sqrt{a^2+b^2} = \sqrt{a^2+c^2} = \sqrt{b^2+c^2} = \sqrt{2} = 1.42 \in W_2$$

However, $\sqrt{a^2+b^2+c^2} = \sqrt{3}$ does not exist. Also, the following condition holds true in W_3

$$\sqrt{a^2+b^2} = \sqrt{a^2+c^2} = \sqrt{b^2+c^2} = \sqrt{2} = 1.416 \in W_3$$

And $\sqrt{a^2+b^2+c^2} = \sqrt{3} = 1.734 \in W_3$ The above theorem implies that this problem has a positive solution in Observer's Mathematics.

3.7 Square Peg Problem

Every continuous simple closed curve in the plane defined by

$$\gamma : S^1 \to R^2$$

contains four points that are the vertices of a square. Is it true or not true? Let's take the Observer's Math point of view. Let's consider the identity $\gamma : S^1 \to S^2$ givne by $\gamma(x, y) = (x, y)$ with $(x, y) \in S^1$. In this case one of vertices (if such square exists) has to be the intersection of line $y = x$ and circle $x^2 +_n y^2 = R^2$. Let's note we assume that circle has a center in $(0, 0)$ and square has edges parallel to coordinate axes. In classical math for any square with vertices (x_0, x_0), $(-x_0, x_0)$, $(x_0, -x_0)$, and $(-x_0, -x_0)$ the circle containing these points always exists.

Theorem 3.13. . *In Observer's Mathematics, for any square with vertices (x_0, x_0), $(-x_0, x_0)$, $(x_0, -x_0)$, and $(-x_0, -x_0)$, the circle containing these points does not always exist.*

Proof. If x_0, $R \in Wn$, $x_0 = 1, 2 = R^2$, then we have

$$n = 2 \to R = 1.42$$

$$n = 3 \to R = 1.416$$

$$n = 4 \to R \text{ does not exist}$$

$$n = 5 \to R = 1.41423$$

$$n = 6 \to R = 1.414216$$

$$n = 7 \to R = 1.4142139$$

$$n = 8 \to R \text{ does not exist}$$

$$n = 9 \to R = 1.414213567$$

$$n = 10 \to R \text{ does not exist}$$

If x_0, $R \in W_n$, $x_0 = 2$, $8 = R^2$, than we have

$$n = 2 \to R = 2.84$$

$$n = 3 \to R \text{ does not exist}$$

$$n = 4 \to R = 2.8287$$

$$n = 5 \to R = 2.82846$$

$$n = 6 \to R \text{ does not exist}$$

$$n = 7 \to R = 2.8284274$$

$$n = 8 \to R = 2.82842717$$

$$n = 9 \to R = 2.828427129$$

$$n = 10 \to R \text{ does not exist}$$

If x_0, $R \in W_n$, $x_0 = 3$, $18 = R^2$, than we have

$$n = 2 \to R \text{ does not exist}$$

$$n = 3 \to R = 4.243$$

$$n = 4 \to R = 4.2427$$

$$n = 5 \to R = 4.24265$$

$$n = 6 \to R \text{ does not exist}$$

$$n = 7 \to R = 4.2426408$$

$$n = 8 \to R = 4.2426407$$

$$n = 9 \to R = 4.242640689$$

$$n = 10 \to R \text{ does not exist}$$

If x_0 , $R \in W_n$, $x_0 = 4$, $32 = R^2$, than we have

$$n = 2 \to R \text{ does not exist}$$

$$n = 3 \to R = 5.658$$

$$n = 4 \to R \text{ does not exist}$$

$$n = 5 \to R \text{ does not exist}$$

$$n = 6 \to R \text{ does not exist}$$

$$n = 7 \to R \text{ does not exist}$$

$$n = 8 \to R \text{ does not exist}$$

$$n = 9 \to R \text{ does not exist}$$

$$n = 10 \to R \text{ does not exist}$$

If x_0 , $R \in W_n$, $x_0 = 5$, $50 = R^2$, than we have

$$n = 2 \to R \text{ does not exist}$$

$$n = 3 \to R \text{ does not exist}$$

$$n = 4 \to R \text{ does not exist}$$

$$n = 5 \to R \text{ does not exist}$$

$$n = 6 \to R \text{ does not exist}$$

$$n = 7 \to R = 7.0710679$$

$$n = 8 \to R \text{ does not exist}$$

$$n = 9 \to R \text{ does not exist}$$

$$n = 10 \to R \text{ does not exist}$$

□

THEOREM 3.14. *In Observer's Mathematics for any circle the square with vertices* (x_0, x_0), $(-x_0, x_0)$, $(x_0, -x_0)$, *and* $(-x_0, -x_0)$ *does not always exist.*

Proof. If x_0, $R \in W_n$, $R = 1$, $2 \times_n x_0^2 = 1$, than we have

$$n = 2 \to x_0 \text{ does not exist}$$

$$n = 3 \to x_0 \text{ does not exist}$$

$$n = 4 \to x_0 \text{ does not exist}$$

$$n = 5 \to x_0 \text{ does not exist}$$

$$n = 6 \to x_0 \text{ does not exist}$$

$$n = 7 \to x_0 \text{ does not exist}$$

$$n = 8 \to x_0 \text{ does not exist}$$

$$n = 9 \to x_0 = 0.70710679*$$

where $*$ means any digit $\in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

$$n = 10 \rightarrow x_0 \text{ does not exist}$$

□

That means that the Square Peg Problem has negative solution in Observer's Mathematics.

3.8 Classical geometric problem of angle trisection

Consider the equation

$$x^3 - 3x - 2\cos\alpha = 0$$

Note that if we take unit circle on the real plane $x^2 + y^2 = 1$ and put $\cos\alpha = z(z \in (0,1))$ for some W_n, then $\sin\alpha$ may not exist. For example, in W_2, if

$$\cos\alpha = 0.6, 0.61, 0.62, 0.63, 0.64, 0.65, 0.66, 0.67, 0.68, 0.69$$

, then

$$\sin\alpha = 0.8, 0.81, 0.82, 0.83, 0.84, 0.85, 0.86, 0.87, 0.88, 0.89$$

though these 10 different $\sin\alpha$ values correspond to each cos a value. Also, If

$$\cos\alpha = 0.8, 0.81, 0.82, 0.83, 0.84, 0.85, 0.86, 0.87, 0.88, 0.89$$

then

$$\sin\alpha = 0.6, 0.61, 0.62, 0.63, 0.64, 0.65, 0.66, 0.67, 0.68, 0.69$$

, though these 10 different $\sin\alpha$ values correspond to each $\cos\alpha$ value. For any other possible positive values of $\cos\alpha$ in W_2 the $\sin\alpha$ does not exist.

THEOREM 3.15. *For any possible positive value of* $\cos\alpha$ *in* W_2 *equation* $x^3 - 3x - 2\cos\alpha = 0$ *does not have a solution in* W_2.

THEOREM 3.16. *For* $\cos\alpha = 0.492 \in W_3$*, in this case*

$$\sin\alpha = 0.88, 0.881, 0.882, 0.883, 0.884, 0.885, 0.886, 0.887, 0.888, 0.889$$

then the solution of equation $x^3 - 3x - 2\cos\alpha = 0$ *exists and it is* $x = 1.88$ *in* W_3. Proof of both theorems follows from direct calculations.

4. PROBABILITY IN QUANTUM THEORY FROM OBSERVER'S MATHEMATICS POINT OF VIEW

Randomness appears in Observer's Mathematics when we consider derivatives. In particular, the following theorem was proven: "From the point of view of a W_m-observer, a derivative calculated by a W_n-observer with $m > n$ is not uniquely defined, i.e., $f'(x_0)$ is a random variable for any real function $f(x)$ on a set of real numbers." In this section we continue to consider the probability questions that appear automatically, without any additional assumptions in Quantum Physics, from Observer's mathematics point of view, see 18 and 19.

Classical Hamilton equations are generally written as follows:

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q}$$
$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}$$

In the above equations, the dot denotes the ordinary derivative with respect to time of the functions $p = p(t)$, called generalized momenta, and $q = q(t)$, called generalized coordinates, taking values in some vector space, and $\mathcal{H} = \mathcal{H}(p, q, t)$ is the so-called Hamiltonian, or (scalar valued) Hamiltonian function. Thus, more explicitly, one can equivalently write

$$\frac{d}{dt}p(t) = -\frac{\partial}{\partial q}\mathcal{H}(p(t), q(t), t)$$

$$\frac{d}{dt}q(t) = -\frac{\partial}{\partial q}\mathcal{H}(p(t), q(t), t)$$

and specify the domain of values in which the parameter t (time) varies.

The simplest interpretation of the Hamilton equations is as follows, applying them to a one-dimensional system consisting of one particle of mass m under time independent boundary conditions: the Hamiltonian $\mathcal{H}$ represents the energy of the system, which is the sum of kinetic and potential energy, traditionally denoted T and V, respectively. Here q is the $x-$coordinate and p is the momentum, mv. Then

$$\mathcal{H} = T + V$$

$$T = \frac{p^2}{2m}$$

$$V = V(q) = V(x)$$

Now, the time-derivative of the momentum p equals the Newtonian force, and so here the first Hamilton equation means that the force on the particle equals the rate at which it loses potential energy with respect to changes in x, its location.

The time-derivative of q here means the velocity: the second Hamilton equation here means that the particle's velocity equals the derivative of its kinetic energy with respect to its momentum. (Because the derivative with respect to p of $p^2/2m$ equals $p/m = mv/m = v$.)

In terms of the generalized coordinates q and generalized velocities $\dot{q}$, we can perform the following steps:

1. Write out the Lagrangian $\mathcal{L} = T - V$. Express T and V as though Lagrange's equation were to be used.

2. Calculate the momenta by differentiating the Lagrangian with respect to velocity (we consider K particles):

$$p_i(q_i, \dot{q}_i, t) = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

3. Express the velocities in terms of the momenta by inverting the expressions in step 2.

4. Calculate the Hamiltonian using the usual definition of $\mathcal{H}$ as the Legendre transformation of $\mathcal{L}$ via

$$\mathcal{H} = \sum_{i=1}^{K} \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} = \sum_{i=1}^{K} \dot{q}_i p_i - \mathcal{L}$$

   Substitute for the velocities using the results in step (3).

5. Apply Hamilton's equations.

We can derive Hamilton's equations by looking at how the total differential of the Lagrangian depends on time, generalized positions q_i and generalized velocities $\dot{q}_i$.

$$d\mathcal{L} = \sum_{i=1}^{K} \left(\frac{\partial \mathcal{L}}{\partial q_i} dq_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t} dt$$

Now the generalized momenta were defined as $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ and Lagrange's equations tell us that

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

We can rearrange this to get

$$\frac{\partial \mathcal{L}}{\partial q_i} = \dot{p}_i$$

and substitute the result into the total differential of the Lagrangian

$$d\mathcal{L} = \sum_{i=1}^{K} (\dot{p}_i dq_i + p_i d\dot{q}_i) + \frac{\partial \mathcal{L}}{\partial t} dt$$

$$d\mathcal{L} = \sum_{i=1}^{K} (\dot{p}_i dq_i + d(p_i \dot{q}_i) - \dot{q}_i dp_i) + \frac{\partial \mathcal{L}}{\partial t} dt$$

and rearrange again to get

$$d\left(\sum_{i=1}^{K} p_i \dot{q}_i - \mathcal{L}\right) = \sum_{i=1}^{K} (-\dot{p}_i dq_i + \dot{q}_i dp_i) - \frac{\partial \mathcal{L}}{\partial t} dt$$

The term on the left-hand side is just the Hamiltonian that we have defined before, so we find that

$$d\mathcal{H} = \sum_{i=1}^{K} (-\dot{p}_i dq_i + \dot{q}_i dp_i) - \frac{\partial \mathcal{L}}{\partial t} dt = \sum_{i=1}^{K} \left[\frac{\partial \mathcal{H}}{\partial q_i} dq_i + \frac{\partial \mathcal{H}}{\partial p_i} dp_i\right] + \frac{\partial \mathcal{H}}{\partial t} dt$$

where the second equality holds because of the definition of the total differential of $\mathcal{H}$ in terms of its partial derivatives. Associating terms from both sides of the equation above yields Hamilton's equations

$$\frac{\partial \mathcal{H}}{\partial q} = -\dot{p}, \frac{\partial \mathcal{H}}{\partial p} = \dot{q}, \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

Starting with Lagrangian mechanics, the equation of motion is based on generalized coordinates q_i and matching generalized velocities $\dot{q}_i$. We write the Lagrangian as

$$\mathcal{L}(q_i, \dot{q}_i, t)$$

with the subscripted variables understood to represent these variables of that type. Hamiltonian mechanics aims to replace the generalized velocity variables with generalized momentum variables, also known as conjugate momenta. By doing so, it is possible to handle certain systems, such as aspects of quantum mechanics, that would otherwise be even more complicated.

For each generalized velocity, there is one corresponding conjugate momentum, defined as:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

The Hamiltonian is the Legendre transform of the Lagrangian:

$$\mathcal{H}(q_i, p_i, t) = \sum_{j=1}^{K} \dot{q}_j p_j - \mathcal{L}(q_i, \dot{q}_i, t)$$

for $i = 1, ..., K$.

If the transformation equations defining the generalized coordinates are independent of t, and the Lagrangian is a product of functions (in the generalized coordinates) which are homogeneous of order 0, 1 or 2, then it can be shown that $\mathcal{H}$ is equal to the total energy $E = T + V$.

Each side in the definition of $\mathcal{H}$ produces differential:

$$d\mathcal{H} = \sum_{i=1}^{K} \left[\frac{\partial \mathcal{H}}{\partial q_i} dq_i + \frac{\partial \mathcal{H}}{\partial p_i} dp_i \right] + \frac{\partial \mathcal{H}}{\partial t} dt =$$

$$= \sum_{i=1}^{K} \left[\dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i \right] - \frac{\partial \mathcal{L}}{\partial t} dt$$

Substituting the previous definition of the conjugate momenta into this equation and matching coefficients, we obtain the equations of motion of Hamiltonian mechanics, known as the canonical equations of Hamilton:

$$\frac{\partial \mathcal{H}}{\partial q_i} = -\dot{p}_i, \frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i, \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

The main relation in classical case is

$$(p + \partial p) \times (\dot{q} + \partial \dot{q}) - p \times \dot{q} = p \times \partial \dot{q} + \dot{q} \times \partial p$$

In Observer's Mathematics in W_n (from $m-$observer point of view with $m > 4n$), the left hand side (LHS) becomes:

$$(p +_n \partial p) \times_n (\dot{q} +_n \partial \dot{q}) -_n p \times_n \dot{q}$$

while the right hand side (RHS) becomes

$$p \times_n \partial \dot{q} +_n \dot{q} \times_n \partial p$$

Crucial difference is that LHS is not always equal to RHS.

Next, we prove the following theorems.

Theorem 4.1. *If $p, \dot{q} \in W_2$, from $m-$observer point of view with $m > 8$, then*

$$P\left((p +_2 \partial p) \times_2 (\dot{q} +_2 \partial \dot{q}) -_2 p \times_2 \dot{q} = p \times_2 \partial \dot{q} +_2 \dot{q} \times_2 \partial p\right) = 0.8$$

where P is the probability.

Proof. If we put $\partial p = \pm 0.01$ and $\partial \dot{q} = \pm 0.01$ and take $p = xy.za$ and $\dot{q} = uv.wb$, $x, y, z, u, v, w \in \{0, 1, \ldots, 9\}$ with $a \neq 9$, $b \neq 9$, (for $\delta p = \delta \dot{q} = 0.01$) and with $a \neq 0$, $b \neq 0$ (for $\delta p = \delta \dot{q} = -0.01$) then we have this identity. But if $a = 0$ or $b = 0$ (for $\delta p = \delta \dot{q} = -0.01$), then this identity becomes wrong. □

THEOREM 4.2. *If $p, \dot{q} \in W_n$, from $m-$observer point of view with $m > 4n$, then*

$$P\left((p +_n \partial p) \times_n (\dot{q} +_n \partial \dot{q}) -_n p \times_n \dot{q} = p \times_n \partial \dot{q} +_n \dot{q} \times_n \partial p\right) = P_{m,n} < 1$$

where $P_{m,n}$ is the probability dependent on m and n.

Proof. Similar to proof of Theorem 4.1. □

THEOREM 4.3. *If $p, \dot{q} \in W_n$, from $m-$observer point of view with $m > 4n$, then*

$$P(d\mathcal{H} \equiv d(p \times_n \dot{q} -_n \mathcal{L}(q, \dot{q}, t)) =$$

$$= \dot{q} \times_n \partial p -_n \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q} +_n \partial \dot{q}, t +_n \partial t) \times_n \partial q -_n \frac{\partial \mathcal{L}}{\partial t}(q, \dot{q} +_n \partial \dot{q}, t) \times_n \partial t) = P_{m,n} < 1$$

Proof. Similar to proof of Theorem 4.1. □

THEOREM 4.4. *If $p, \dot{q} \in W_n$, from $m-$observer point of view with $m > 4n$, then*

$$P(d\mathcal{H} \equiv d(p \times_n \dot{q} -_n \mathcal{L}(q, \dot{q}, t)) =$$

$$= \dot{q} \times_n \partial p -_n \frac{\partial \mathcal{L}}{\partial q}(q, \dot{q}, t) \times_n \partial q -_n \frac{\partial \mathcal{L}}{\partial t}(q, \dot{q}, t) \times_n \partial t) = P_{m,n,\mathcal{L}} < 1$$

where $P_{m,n,\mathcal{L}}$ is the probability dependent on m, n, and $\mathcal{L}$.

Proof. Similar to proof of Theorem 4.1. □

THEOREM 4.5. *($K-$bodies solution) If $p, \dot{q} \in W_n$ from $m-$observer point of view with $m \geq log_{10}\left((2 \times 10^{2n} - 1)^{2k} + 1\right)$ then*

$$P\left(d\mathcal{H} = \sum_{i=1}^{K}(\dot{q}_i \times_n \partial p_i -_n \frac{\partial \mathcal{L}}{\partial q_i}(q_i, \dot{q}_i, t) \times_n \partial q_i -_n \frac{\partial \mathcal{L}}{\partial t}(q_i, \dot{q}_i, t) \times_n \partial t\right) = P_{m,n,\mathcal{L},K} < 1$$

where $P_{m,n,\mathcal{L},K}$ is the probability dependent on m, n, $\mathcal{L}$, and K.

Proof. Similar to proof of Theorem 4.1. □

Let's now consider several examples. For all the tables below, we let $p_i \in [P_{imin}, P_{imax})$ and $q_i \in [Q_{imin}, Q_{imax})$. Also, "NaN" means that at least one of $(p +_n \partial p) \times_n (\dot{q} +_n \partial \dot{q})$, $p \times_n \dot{q}$, or $(p +_n \partial p) \times_n (\dot{q} +_n \partial \dot{q}) -_n p \times_n \dot{q}$ does not belong to W_n, "eq" means number of pairs $(p, \dot{q})$ with LHS = RHS, "not eq" means number of pairs $(p, \dot{q})$ with LHS≠RHS, and "probability"= $\frac{eq}{eq+noteq}$. Also, each of the examples provided below will be given for three (top table) and five (bottom table) bodies.

For the first example, we let $n = 2$, i.e., granularity= 2, with the min and max p and q values provided in the table.

Version	1.6					
Granularity	2					
	Pmin	Pmax	dP	Qmin	Qmax	dQ
	0.01	1	0.01	0.01	99.98	0.01
	0.01	1	0.01	0.01	99.98	0.01
	0.01	1	0.01	0.01	99.98	0.01
	NaN	eq	not eq	probability		
Fast version:	0	5.14E+17	4.56E+17	0.52977727643216		
	Pmin	Pmax	dP	Qmin	Qmax	dQ
	0.01	1	0.01	0.01	99.98	0.01
	0.01	1	0.01	0.01	99.98	0.01
	0.01	1	0.01	0.01	99.98	0.01
	0.01	1	0.01	0.01	99.98	0.01
	0.01	1	0.01	0.01	99.98	0.01
	NaN	eq	not eq	probability		
Fast version:	0	3.29E+29	6.20E+29	0.34686105751432		

For the second example, we let $n = 4$, i.e., granularity$= 4$, with the min and max p and q values provided in the table.

Version	1.6					
Granularity	4					
	Pmin	Pmax	dP	Qmin	Qmax	dQ
	0.0001	1	0.0001	0.0001	9999.9998	0.0001
	0.0001	1	0.0001	0.0001	9999.9998	0.0001
	0.0001	1	0.0001	0.0001	9999.9998	0.0001
	NaN	eq	not eq	probability		
Fast version:	0	5.31E+35	4.68E+35	0.5314232961251		
	Pmin	Pmax	dP	Qmin	Qmax	dQ
	0.0001	1	0.0001	0.0001	9999.9998	0.0001
	0.0001	1	0.0001	0.0001	9999.9998	0.0001
	0.0001	1	0.0001	0.0001	9999.9998	0.0001
	0.0001	1	0.0001	0.0001	9999.9998	0.0001
	0.0001	1	0.0001	0.0001	9999.9998	0.0001
	NaN	eq	not eq	probability		
Fast version:	0	3.48E+59	6.51E+59	0.34865908112777		

For the third example, we let $n = 2$, i.e., granularity$= 2$, with the min and max p and q values provided in the table.

Version	1.6
Granularity	2

Pmin	Pmax	dP	Qmin	Qmax	dQ
0.01	3	0.01	0.01	33.32	0.01
0.01	3	0.01	0.01	33.32	0.01
0.01	3	0.01	0.01	33.32	0.01

	NaN	eq	not eq	probability
Fast version:	0	5.25E+17	4.63E+17	0.53090187908807

Pmin	Pmax	dP	Qmin	Qmax	dQ
0.01	3	0.01	0.01	33.32	0.01
0.01	3	0.01	0.01	33.32	0.01
0.01	3	0.01	0.01	33.32	0.01
0.01	3	0.01	0.01	33.32	0.01
0.01	3	0.01	0.01	33.32	0.01

	NaN	eq	not eq	probability
Fast version:	0	3.41E+29	6.39E+29	0.34808911075464

For the fourth example, we let $n = 4$, i.e., granularity$= 4$, with the min and max p and q values provided in the table.

Version	1.6
Granularity	4

Pmin	Pmax	dP	Qmin	Qmax	dQ
0.0001	3	0.0001	0.0001	3333.3332	0.0001
0.0001	3	0.0001	0.0001	3333.3332	0.0001
0.0001	3	0.0001	0.0001	3333.3332	0.0001

	NaN	eq	not eq	probability
Fast version:	0	5.31E+35	4.69E+35	0.53143510023939

Pmin	Pmax	dP	Qmin	Qmax	dQ
0.0001	3	0.0001	0.0001	3333.3332	0.0001
0.0001	3	0.0001	0.0001	3333.3332	0.0001
0.0001	3	0.0001	0.0001	3333.3332	0.0001
0.0001	3	0.0001	0.0001	3333.3332	0.0001
0.0001	3	0.0001	0.0001	3333.3332	0.0001

	NaN	eq	not eq	probability
Fast version:	0	3.49E+59	6.51E+59	0.34867198873564

For the fifth example, we let $n = 2$, i.e., granularity$= 2$, with the min and max p and q values provided in the table.

Version 1.6

Granularity 2

Pmin	Pmax	dP	Qmin	Qmax	dQ
0.01	5	0.01	0.01	19.98	0.01
0.01	5	0.01	0.01	19.98	0.01
0.01	5	0.01	0.01	19.98	0.01

	NaN	eq	not eq	probability
Fast version:	0	5.26E+17	4.63E+17	0.53170684796328

Pmin	Pmax	dP	Qmin	Qmax	dQ
0.01	5	0.01	0.01	19.98	0.01
0.01	5	0.01	0.01	19.98	0.01
0.01	5	0.01	0.01	19.98	0.01
0.01	5	0.01	0.01	19.98	0.01
0.01	5	0.01	0.01	19.98	0.01

	NaN	eq	not eq	probability
Fast version:	0	3.43E+29	6.40E+29	0.34896919331919

For the sixth example, we let $n = 4$, i.e., granularity$= 4$, with the min and max p and q values provided in the table.

Version 1.6

Granularity 4

Pmin	Pmax	dP	Qmin	Qmax	dQ
0.0001	5	0.0001	0.0001	1999.9998	0.0001
0.0001	5	0.0001	0.0001	1999.9998	0.0001
0.0001	5	0.0001	0.0001	1999.9998	0.0001

	NaN	eq	not eq	probability
Fast version:	0	5.31E+35	4.69E+35	0.53143751899806

Pmin	Pmax	dP	Qmin	Qmax	dQ
0.0001	5	0.0001	0.0001	1999.9998	0.0001
0.0001	5	0.0001	0.0001	1999.9998	0.0001
0.0001	5	0.0001	0.0001	1999.9998	0.0001
0.0001	5	0.0001	0.0001	1999.9998	0.0001
0.0001	5	0.0001	0.0001	1999.9998	0.0001

	NaN	eq	not eq	probability
Fast version:	0	3.49E+59	6.51E+59	0.34867463363269

For the seventh example, we let $n = 2$, i.e., granularity$= 2$, with the min and max p and q values provided in the table.

Version	1.6					
Granularity	2					

Pmin	Pmax	dP	Qmin	Qmax	dQ
0.01	10	0.01	0.01	9.98	0.01
0.01	10	0.01	0.01	9.98	0.01
0.01	10	0.01	0.01	9.98	0.01

	NaN	eq	not eq	probability
Fast version:	0	5.26E+17	4.62E+17	0.53250801093107

Pmin	Pmax	dP	Qmin	Qmax	dQ
0.01	10	0.01	0.01	9.98	0.01
0.01	10	0.01	0.01	9.98	0.01
0.01	10	0.01	0.01	9.98	0.01
0.01	10	0.01	0.01	9.98	0.01
0.01	10	0.01	0.01	9.98	0.01

	NaN	eq	not eq	probability
Fast version:	0	3.43E+29	6.37E+29	0.34984599725169

Finally, for the eighth example, we let $n = 4$, i.e., granularity$= 4$, with the min and max p and q values provided in the table.

Version	1.6					
Granularity	4					

Pmin	Pmax	dP	Qmin	Qmax	dQ
0.0001	10	0.0001	0.0001	999.9998	0.0001
0.0001	10	0.0001	0.0001	999.9998	0.0001
0.0001	10	0.0001	0.0001	999.9998	0.0001

	NaN	eq	not eq	probability
Fast version:	0	5.31E+35	4.69E+35	0.53143935251679

Pmin	Pmax	dP	Qmin	Qmax	dQ
0.0001	10	0.0001	0.0001	999.9998	0.0001
0.0001	10	0.0001	0.0001	999.9998	0.0001
0.0001	10	0.0001	0.0001	999.9998	0.0001
0.0001	10	0.0001	0.0001	999.9998	0.0001
0.0001	10	0.0001	0.0001	999.9998	0.0001

	NaN	eq	not eq	probability
Fast version:	0	3.49E+59	6.51E+59	0.34867663857897

Now we formulate additional theorems. Let probability that LHS = RHS be $P_{n,m,K}$, where the situation is from $m-$Observer's point of view, while the calculations are in W_n, with $m > 4nK$, and K is the number of particles. Then we have the following theorems.

THEOREM 4.6. *$P_{n,m,K} \to 0$ when $K \to \infty$, with m, n are the same from $m-$Observer's point of view*

Proof. The proof is similar to proofs of theorems 4.1 - 4.5. □

THEOREM 4.7. *Let $p \in [a_1, b_1]$, $\dot{q} \in [c_1, d_1]$ and $[a_1, b_1] \times_n [c_1, d_1] = W_n$. With these conditions, let probability that LHS = RHS be $P_{n,m,K,[a_1,b_1],[c_1,d_1]}$. Also, let $p \in [a_2, b_2]$, $\dot{q} \in [c_2, d_2]$ and $[a_1, b_1] \times_n [c_1, d_1] = W_n$. With these conditions, let probability that LHS = RHS be $P_{n,m,K,[a_2,b_2],[c_2,d_2]}$. Then $P_{n,m,K,[a_1,b_1],[c_1,d_1]} = P_{n,m,K,[a_2,b_2],[c_2,d_2]}$*

Proof. The proof is similar to proofs of theorems 4.1 - 4.5. □

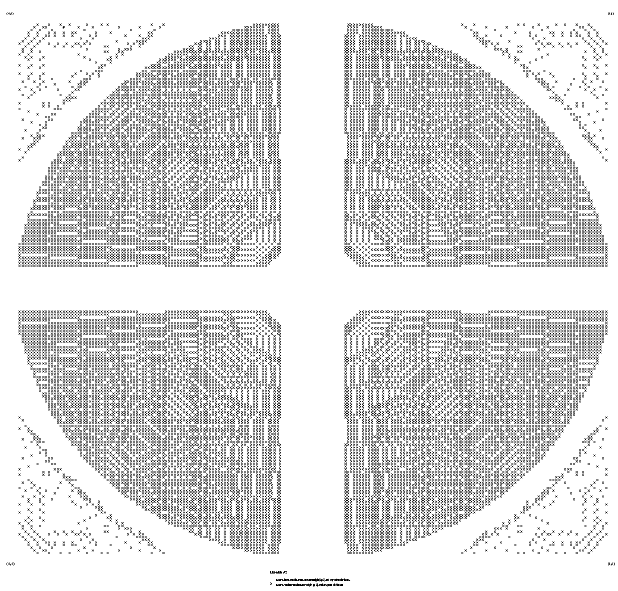
THEOREM 4.8. *$P_{n_1,m,K} = P_{n_2,m,K}$ with m, K are the same from $m-$Observer's point of view and $m > 4Kn_1$ and $m > 4Kn_2$*

Proof. The proof is similar to proofs of theorems 4.1 - 4.5. □

5. NADEZHDA EFFECT

In this section we consider an open square Q centered at the origin with sides of length 2 located on a plane $W_n \times W_n$. We will calculate the distance D between the origin $(0,0)$ and any point of Q as follows. $D = \rho((0,0),(x,y)) = \sqrt{x^2+y^2} = \sqrt{x \times_n x +_n y \times_n y}$, where $\sqrt{a} = b$ means $b \times_n b = a$, $x, y \in Q$, i.e., $|x| < 1$, $|y| < 1$.

The figure below (see 20) contains an illustration of the fact that for some points on $W_n \times W_n$ the concept of distance from the origin does not exist; while for others it does exist. The illustration below is for $n = 3$ $(Q \subset W_3 \times W_3)$. Points with no distance to the origin are indicated by black, while points where distance from the origin exists are indicated in white.



This means that the distance D does not always exist, i.e., not every segment on a plane has a length. This phenomenon occurs for all n. We call the presence of these "black holes" as the Nadezhda Effect. This effect gives us new possibilities for discovering physical processes and developing their mathematical models.

Theorem 5.1. **Nadezhda Effect Theorem.** *For any positive integer n and W_n, consider the plane $W_n \times W_n = \{(x, y)\}, x, y \in W_n$ with standard Euclidean metric $d^2((x_1, y_1), (x_2, y_2)) = (x_1 - x_2)^2 + (y_1 - y_2)^2$. Next, consider any line $y = k \times_n x$, with $y, k, x \in W_n$. Then there is some point $(x_0, y_0) = (x_0, k \times_n x_0) \in W_n \times W_n$ such that $d((x_0, y_0), (0, 0))$ does not exist.*

Proof. First, let's consider several samples. Let's take $y = k \times_2 x$, $k, x \in W_2$, $k = 1, 2, \ldots, 10$. Also, let distance $\rho = \sqrt{x \times_2 x + (k \times_2 x) \times_2 (k \times_2 x)}$, with $x \in W_2$. Let's take $x \in [0, 0.09]$. In these cases, ρ exists. If we take $x = 1$, ρ does not exist for all $k = 1, 2, \ldots, 10$. □

Similarly, we can prove the following lemmas.

Lemma 1. If $y = 0.\underbrace{0...01}_{n} \times_n x$, then there exists $x_0 \in W_n$ such that ρ does not exist, where

$$\rho = \sqrt{x_0 \times_n x_0 + (0.\underbrace{0...01}_{n} \times_n x_0) \times_n (0.\underbrace{0...01}_{n} \times_n x_0)}$$

.

Lemma 2. If $y = k \times_n x$ with $0 \le k \le 1$, then there exists $x_0 \in W_n$ such that ρ does not exist, where

$$\rho = \sqrt{x_0 \times_n x_0 + (k \times_n x_0) \times_n (k \times_n x_0)}$$

.

Lemma 3. If $y = \underbrace{99...9}_{n}.\underbrace{99...9}_{n} \times_n x$, then there exists $x_0 \in W_n$ such that ρ does not exist, where

$$\rho = \sqrt{x_0 \times_n x_0 + (\underbrace{99...9}_{n}.\underbrace{99...9}_{n} \times_n x_0) \times_n (\underbrace{99...9}_{n}.\underbrace{99...9}_{n} \times_n x_0)}$$

.

Lemma 4. If $y = k \times_n x$ with $1 < k \leq \underbrace{99...9}_{n}.\underbrace{99...9}_{n}$, then there exists $x_0 \in W_n$ such that ρ does not exist, where

$$\rho = \sqrt{x_0 \times_n x_0 + (k \times_n x_0) \times_n (k \times_n x_0)}$$

.

Proof of Theorem follows from Lemmas 1 - 4.

6. LAGRANGIAN IN CLASSICAL MECHANICS AND SPECIAL RELATIVITY FROM OBSERVER'S MATHEMATICS POINT OF VIEW

It is well known that Bohr's position and the Copenhagen interpretation of Quantum Mechanics (QM) results in an observer-based viewpoint to physics. Another major aspect of QM is the stipulation of discretization of space-time instead of its classical continuity interpretation, which is explicitly related to the Heisenberg uncertainty principle. Heisenberg's unique contribution was not to point out that measurement affects the system being measured, but rather, it was to recognize the new fundamental limits to measurement set by the "quantum of actions". There are two such limits. First, according to classical physics we can make the disturbance as small as we wish, while according to quantum mechanics, we cannot. The action of light, for instance, is quantized, so that a photon cannot avoid disturbing a particle it strikes. The second limit imposed by quantum mechanics is that this disturbance is uncontrollable and unpredictable. This latter feature reflects the deeply statistical nature of quantum mechanics. The two new features appearing in Heisenberg's analysis, therefore, are:

- The disturbance cannot be reduced in magnitude below a fundamental limit, and
- Correction for the disturbance is impossible.

Randomness appeared in Observer's Mathematics when we considered derivatives. In particular, the following theorem was proven: "From the point of view of a W_m−observer, a derivative calculated by a W_n-observer with $m > n$ is not uniquely defined, i.e., $f'(x_0)$ is a random variable for any real function $f(x)$ on a set of real numbers." In this section we continue to consider the probability questions that appear automatically, without any additional assumptions in Quantum Physics, from Observer's mathematics point of view.

We also see that randomness appears here not only when we consider derivatives, but also in elementary arithmetic calculations.

Let's consider the Lagrangian for a free particle in classical mechanics. The following discussion is based on 21. Consider the simplest case, that of the free motion of a particle relative to an inertial frame of reference. The Lagrangian in this case can depend only on the square of the velocity. To discover the form of this dependence, we make use of Galileo's relativity principle. If an inertial frame K is moving with an infinitesimal velocity ε relative to another inertial frame K', then $\mathbf{v}' = \mathbf{v} + \varepsilon$. Since the equations of motion must have the same form in every frame, the Lagrangian $L(v^2)$ must be converted by this transformation into a function L' which differs from $L(v^2)$, if at all, only by the total time derivative of a function of coordinates and time.

We have $L' = L(v'^2) = L\left(v^2 + 2\mathbf{v} \cdot \varepsilon + \varepsilon^2\right)$. Expanding this expression in powers of ε and neglecting terms above the first order, we obtain

$$L(v'^2) = L(v^2) + \frac{\partial L}{\partial v^2} 2\mathbf{v} \cdot \varepsilon$$

The second term on the right of this equation is a total time derivative only if it a linear function of the velocity $\mathbf{v}$. Hence $\frac{\partial L}{\partial v^2}$ is independent of the velocity, i.e., the Lagrangian is in this case proportional to the square of the velocity, and we write it as

$$L = \frac{1}{2}mv^2$$

From the fact that a Lagrangian of this form satisfies Galileo's relativity principle for an infinitesimal relative velocity, it follows at once that the Lagrangian is invariant for a finite relative velocity $\mathbf{V}$ of the frames K and K'. For

$$L' = \frac{1}{2}mv'^2 = \frac{1}{2}m\left(\mathbf{v} + \mathbf{V}\right)^2 = \frac{1}{2}mv^2 + m\mathbf{v}\cdot\mathbf{V} + \frac{1}{2}m\mathbf{V}^2$$

or

$$L' = L + \frac{d\left(m\mathbf{v}\cdot\mathbf{V} + \frac{1}{2}\mathbf{V}^2 t\right)}{dt}$$

The second term is a total time derivative and may be omitted.

Let's consider the Lagrangian for a free particle in special relativity. The following discussion is based on.[22] The principle of Least Action states that a mechanical system should have a quantity called the action S. Such quantity is minimized (in other words, $\delta S = 0$ for the actual motion of the system. The action of a relativistic system should be

1. a scalar, that means Lorentz transformations will not affect this quantity,

2. an integral of which the integrand is a first-order differential.

The only quantity that satisfies the two criteria above is the space-time interval ds, or a scalar multiple thereof. In short, we can conclude that the action must have the following form: $S = \kappa \int ds$. We have

$$ds = \sqrt{c^2 dt - dx^2 - dy^2 - dz^2}$$

After pulling out cdt from the square root and noting that $\frac{dx^2+dy^2+dz^2}{dt^2} = \upsilon^2$, we have $c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt^2 - \upsilon^2dt^2 = (c^2 - \upsilon^2)\, dt$ and thus

$$ds = cdt\sqrt{1 - \frac{\upsilon^2}{c^2}}$$

Hence

$$S = c\kappa \int \sqrt{1 - \frac{\upsilon^2}{c^2}} dt$$

Now, the action integral can be expressed as a time integral of the Lagrangian between two fixed times:

$$S = \int L dt$$

Then we can just read off the Lagrangian:

$$L = c\kappa\sqrt{1 - \frac{\upsilon^2}{c^2}}$$

What is remaining now is determining the expression for κ. At this point we should note that for low velocity υ, this relativistic expression for the Lagrangian should resemble that of the classical free Lagrangian $L = \frac{1}{2}m\upsilon^2$. To compare the two Lagrangians, we perform a Taylor expansion on the square root:

$$L = c\kappa \left(1 - \frac{\upsilon^2}{2c^2} + O(\upsilon^4)\right)$$

The firs term, $c\kappa$, is a constant. That will not affect the equations of motion (for example, Euler-Lagrange Equation). The second term, after expanding out, is equal to $-\kappa\frac{\upsilon^2}{2c}$. To reduce to the classical limit, we can put $\kappa = -mc$. Therefore, the relativistic Lagrangian is:

$$L = -mc^2\sqrt{1 - \frac{\upsilon^2}{c^2}}$$

Let us consider the Observer's Mathematics point of view, see 4 and 23. Note, that in the calculations above, we used two fundamental arithmetic formulas that use distributive property of real numbers: $(a+b)^2 = a^2 + 2ab + b^2$ and $c(a+b) = ca + cb$. In Observer's Mathematics, we need to re-write the first formula as follows:

$$(a +_n b) \times_n (a +_n b) = (a \times_n a +_n 2 \times_n (a \times_n b)) +_n b \times_n b$$

If we now recall theorems 2.4, 2.5 and 2.6, then we have proved the following theorems.

THEOREM 6.1. *In classical mechanics,* $P\left(L = \frac{mv^2}{2}\right) < 1$, *where* P *is the probability.*

THEOREM 6.2. *In special relativity,* $P\left(L = -mc^2\sqrt{1 - \frac{v^2}{c^2}}\right) < 1$, *where* P *is the probability.*

7. PHOTOELECTRIC EFFECT FROM OBSERVER'S MATHEMATICS POINT OF VIEW

In 1922, Albert Einstein received the Nobel Prize - not for his relativity theory, but for his interpretation of the photoelectric effect as being due to particle-like photons striking the surfaces of metals and ejecting electrons. But ironically it has been cogently argued that Einstein's conclusions were not fully justified. The theory of Lamb and Scully (see 24) treated atoms quantum- mechanically, but regarded light as being a purely classical electromagnetic wave with no particle properties. Their conclusion was that the photoelectric effect does not constitute proof of the existence of photons.

Experimenters, therefore, led to design an experiment that asks whether light can be in two different places at the same time. The method is to place two detectors at widely separated locations, illuminate them both with the same light source, and ask whether they click at the same instant. Within the particle picture of light, they should not. The experimental apparatus required for such an experiment has to include: a light source, a half-silvered mirror and two detectors. Light falls on the half-silvered mirror, which acts as a beam splitter. If the incident light intensity is I, then behind the mirror the detectors each register an intensity $\frac{I}{2}$. Each detector responds with "click". Experimenters correlate these clicks by connecting them to a coincidence counter, which records a count only if both detectors click at the same moment. The results of such an experiment are conveniently analyzed in terms of the so-called anticorrelation parameter A:

$$A = \frac{P_c}{P_1 P_2}$$

where P_1 is the experimentally measured probability of detector 1 responding, P_2 is the experimentally measured probability of detector 2 responding, and P_c the probability of coincidence. The quantity A has several properties that make it a particularly useful diagnostic in this situation. On the one hand, if light is composed of photons, the two detectors should never respond together, making P_c zero, so that A should be zero. If, on the other hand, light has no particle-like properties, the detectors should be perfectly capable of clicking together, and A can be non-zero. Indeed, if the detectors turn out to click randomly and independently of one another, experimenters can easily show that A will equal 1. Finally, a measured value of A greater than 1 would show the two detectors to be clicking together more often than purely random behavior would allow a "clustering" tendency of the clicks.

The Hanbury-Brown and Twiss (see 25) experiment was done using this idea. And they used for anticorrelation parameter A calculation of the following formula:

$$A = \frac{\ll I^2 \gg}{\ll I \gg^2}$$

where $\ll I \gg$ is the average intensity over many instantaneous measurements, and $\ll I^2 \gg$ is the average of the intensity squared, see 26. The result shows that the expected anticorrelation parameter within the semi-classical theory (Lamb and Scully, Hanbury-Brown and Twiss) is simply the average of I squared as compared to the square of the average of I. And it was very easy to show that always $A > 1$. To see how that was done, begin with the simple case of a beam whose intensity fluctuates between only two values, I_1 and I_2. Defining x to be the ratio $\frac{I_2}{I_1}$, the averages are

$$\ll I^2 \gg = \frac{1}{2}(I_1^2 + I_2^2) = \frac{1}{2}(I_1^2)(1 + x^2)$$

and

$$\ll I \gg^2 = \left(\frac{1}{2}(I_1 + I_2)\right)^2 = I_1^2 \left(\frac{(1+x)}{2}\right)^2$$

But

$$\frac{1+x^2}{2} \geq \left(\frac{1+x}{2}\right)^2$$

because

$$2(1+x^2) \geq (1+x)^2$$

and

$$(1-x)^2 \geq 0$$

This result can be extended to a beam whose intensity fluctuates between any number of values by using Cauchy-Schwartz inequality:

$$\ll I^2 \gg \geq \ll I \gg^2$$

We now have the following, see 27.

Theorem 7.1. *There are some values of light intensity where anticorrelation parameter $A \in [0, 1)$.*

Proof. For proof it is enough to find a corresponding example. Let's take $n = 2$, $I_1 = 0.2$, and $I_2 = 0.1$. We then have the following:

$$A \times_2 ((0.5 \times_2 (0.2 +_2 0.1)) \times_2 (0.5 \times_2 (0.2 +_2 0.1))) = 0.5 \times_2 ((0.2 \times_2 0.2) +_2 (0.1 \times_2 0.1))$$

which leads to

$$A \times_2 (0.15 \times_2 0.15) = 0.5 \times_2 (0.04 +_2 0.01)$$

which leads to

$$A \times_2 0.01 = 0$$

i.e. $A \in [0, 1)$. □

THEOREM 7.2. *There are some values of light intensity where anticorrelation parameter $A = 1$.*

Proof. For proof it is enough to find corresponding example. Let's take $n = 2$, $I_1 = 1.01$, and $I_2 = 1.02$. We then have the following:

$$A \times_2 ((0.5 \times_2 (1.01 +_2 1.02)) \times_2 (0.5 \times_2 (1.01 +_2 1.02)) = 0.5 \times_2 ((1.01 \times_2 1.01) +_2 (1.02 \times_2 1.02))$$

which leads to

$$A \times_2 (1 \times_2 1) = 0.5 \times_2 (1.02 +_2 1.04)$$

which leads to

$$A \times_2 1 = 1$$

i.e. $A = 1$. □

THEOREM 7.3. *There are some values of light intensity where anticorrelation parameter $A > 1$.*

Proof. For proof it is enough to find corresponding example. Let's take $n = 2$, $I_1 = 1.11$, $I_2 = 1.08$. We then have the following:

$$A \times_2 ((0.5 \times_2 (1.11 +_2 1.08)) \times_2 (0.5 \times_2 (1.11 +_2 1.08)) = 0.5 \times_2 ((1.11 \times_2 1.11) +_2 (1.08 \times_2 1.08))$$

which leads to

$$A \times_2 (1.05 \times_2 1.05) = 0.5 \times_2 (1.23 +_2 1.16)$$

which leads to

$$A \times_2 1.1 = 1.15$$

i.e. $A = 1.05$. □

Theorems 7.1, 7.2, and 7.3 show that with enough small intensities Einstein interpretation of the photoelectric effect as being due to particle-like photons striking the surfaces of metals and ejecting electrons is correct.

8. DIRAC EQUATION FOR FREE ELECTRON

Let's consider Dirac equations for free electron in classical mathematics, see.[28]

$$-m_0 c\psi_2 = \hbar\left(\frac{\partial\psi_{\hat{1}}}{\partial x^3} + \frac{\partial\psi_{\hat{1}}}{\partial x^0} + \frac{\partial\psi_{\hat{2}}}{\partial x^1} + i\frac{\partial\psi_{\hat{2}}}{\partial x^2}\right)$$

$$m_0 c\psi_1 = \hbar\left(\frac{\partial\psi_{\hat{1}}}{\partial x^1} - i\frac{\partial\psi_{\hat{1}}}{\partial x^2} - \frac{\partial\psi_{\hat{2}}}{\partial x^3} + \frac{\partial\psi_{\hat{2}}}{\partial x^0}\right)$$

$$-m_0 c\psi_{\hat{2}} = \hbar\left(\frac{\partial\psi_1}{\partial x^3} + \frac{\partial\psi_1}{\partial x^0} + \frac{\partial\psi_2}{\partial x^1} - i\frac{\partial\psi_2}{\partial x^2}\right)$$

$$m_0 c\psi_{\hat{1}} = \hbar\left(\frac{\partial\psi_1}{\partial x^1} + i\frac{\partial\psi_1}{\partial x^2} - \frac{\partial\psi_2}{\partial x^3} + \frac{\partial\psi_2}{\partial x^0}\right)$$

where $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$, $\hbar = \frac{h}{2\pi}$, h is the Planck Constant, and $\hbar = 1.054\ldots \times 10^{-34}$ m^{2}kg/s, c is the speed of light, and $\psi_1, \psi_2, \psi_{\hat{1}}, \psi_{\hat{2}}$ are the spinors.

Now, consider the same equations in Observer's Mathematics, see 29. $\Psi_1, \Psi_2 \in W_n$. Put $\psi_1 = \psi_1^a + i\psi_1^b$, $\psi_2 = \psi_2^a + i\psi_2^b$, $\psi_{\hat{1}} = \psi_{\hat{1}}^a + i\psi_{\hat{1}}^b$, and $\psi_{\hat{2}} = \psi_{\hat{2}}^a + i\psi_{\hat{2}}^b$.

After that, we have the following eight equations:

1. $-(m_0 \times_n c) \times_n \psi_2^a = \hbar \times_n \left(\left(\left(\frac{\partial\psi_{\hat{1}}^a}{\partial x^3} +_n \frac{\partial\psi_{\hat{1}}^a}{\partial x^0}\right) +_n \frac{\partial\psi_{\hat{2}}^a}{\partial x^1}\right) -_n \frac{\partial\psi_{\hat{2}}^b}{\partial x^2}\right)$

2. $-(m_0 \times_n c) \times_n \psi_2^b = \hbar \times_n \left(\left(\left(\frac{\partial\psi_{\hat{1}}^b}{\partial x^3} +_n \frac{\partial\psi_{\hat{1}}^b}{\partial x^0}\right) +_n \frac{\partial\psi_{\hat{2}}^b}{\partial x^1}\right) +_n \frac{\partial\psi_{\hat{2}}^b}{\partial x^3}\right)$

3. $(m_0 \times_n c) \times_n \psi_1^a = \hbar \times_n \left(\left(\left(\frac{\partial\psi_{\hat{1}}^a}{\partial x^1} +_n \frac{\partial\psi_{\hat{1}}^b}{\partial x^2}\right) -_n \frac{\partial\psi_{\hat{2}}^a}{\partial x^3}\right) +_n \frac{\partial\psi_{\hat{2}}^a}{\partial x^0}\right)$

4. $(m_0 \times_n c) \times_n \psi_1^b = \hbar \times_n \left(\left(\left(\frac{\partial\psi_{\hat{1}}^b}{\partial x^1} -_n \frac{\partial\psi_{\hat{1}}^a}{\partial x^2}\right) -_n \frac{\partial\psi_{\hat{2}}^b}{\partial x^3}\right) +_n \frac{\partial\psi_{\hat{2}}^b}{\partial x^0}\right)$

5. $-(m_0 \times_n c) \times_n \psi_{\hat{2}}^a = \hbar \times_n \left(\left(\left(\frac{\partial\psi_1^a}{\partial x^3} +_n \frac{\partial\psi_1^a}{\partial x^0}\right) +_n \frac{\partial\psi_2^a}{\partial x^1}\right) +_n \frac{\partial\psi_2^b}{\partial x^2}\right)$

6. $-(m_0 \times_n c) \times_n \psi_{\hat{2}}^b = \hbar \times_n \left(\left(\left(\frac{\partial\psi_1^b}{\partial x^3} +_n \frac{\partial\psi_1^b}{\partial x^0}\right) +_n \frac{\partial\psi_2^b}{\partial x^1}\right) -_n \frac{\partial\psi_2^b}{\partial x^2}\right)$

7. $(m_0 \times_n c) \times_n \psi_{\hat{1}}^a = \hbar \times_n \left(\left(\left(\frac{\partial\psi_1^a}{\partial x^1} -_n \frac{\partial\psi_1^b}{\partial x^2}\right) -_n \frac{\partial\psi_2^a}{\partial x^3}\right) +_n \frac{\partial\psi_2^a}{\partial x^0}\right)$

8. $(m_0 \times_n c) \times_n \psi^b_{\hat{1}} = \hbar \times_n \left(\left(\left(\frac{\partial \psi^b_1}{\partial x^1} +_n \frac{\partial \psi^a_1}{\partial x^2}\right) -_n \frac{\partial \psi^b_2}{\partial x^3}\right) +_n \frac{\partial \psi^b_2}{\partial x^0}\right)$

We now have the following theorems.

THEOREM 8.1. *If m_0 is small enough such that $m_0 \times_n c = 0$ and $n > 35$ then*

$$\left(\left(\frac{\partial \psi^a_{\hat{1}}}{\partial x^3} +_n \frac{\partial \psi^a_{\hat{1}}}{\partial x^0}\right) +_n \frac{\partial \psi^a_{\hat{2}}}{\partial x^1}\right) -_n \frac{\partial \psi^b_{\hat{2}}}{\partial x^2} = 0.\overbrace{\underbrace{0\ldots0}_{n-35}*\ldots*}^{n}$$

$$\left(\left(\frac{\partial \psi^b_{\hat{1}}}{\partial x^3} +_n \frac{\partial \psi^b_{\hat{1}}}{\partial x^0}\right) +_n \frac{\partial \psi^b_{\hat{2}}}{\partial x^1}\right) +_n \frac{\partial \psi^b_{\hat{2}}}{\partial x^3} = 0.\overbrace{\underbrace{0\ldots0}_{n-35}*\ldots*}^{n}$$

$$\left(\left(\frac{\partial \psi^a_{\hat{1}}}{\partial x^1} +_n \frac{\partial \psi^b_{\hat{1}}}{\partial x^2}\right) -_n \frac{\partial \psi^a_{\hat{2}}}{\partial x^3}\right) +_n \frac{\partial \psi^b_{\hat{2}}}{\partial x^0} = 0.\overbrace{\underbrace{0\ldots0}_{n-35}*\ldots*}^{n}$$

$$\left(\left(\frac{\partial \psi^b_{\hat{1}}}{\partial x^1} -_n \frac{\partial \psi^b_{\hat{1}}}{\partial x^2}\right) -_n \frac{\partial \psi^b_{\hat{2}}}{\partial x^3}\right) +_n \frac{\partial \psi^b_{\hat{2}}}{\partial x^0} = 0.\overbrace{\underbrace{0\ldots0}_{n-35}*\ldots*}^{n}$$

$$\left(\left(\frac{\partial \psi^a_1}{\partial x^3} +_n \frac{\partial \psi^a_1}{\partial x^0}\right) +_n \frac{\partial \psi^a_2}{\partial x^1}\right) +_n \frac{\partial \psi^b_2}{\partial x^2} = 0.\overbrace{\underbrace{0\ldots0}_{n-35}*\ldots*}^{n}$$

$$\left(\left(\frac{\partial \psi^b_1}{\partial x^3} +_n \frac{\partial \psi^b_1}{\partial x^0}\right) +_n \frac{\partial \psi^b_2}{\partial x^1}\right) -_n \frac{\partial \psi^b_2}{\partial x^2} = 0.\overbrace{\underbrace{0\ldots0}_{n-35}*\ldots*}^{n}$$

$$\left(\left(\frac{\partial \psi^a_1}{\partial x^1} -_n \frac{\partial \psi^b_1}{\partial x^2}\right) -_n \frac{\partial \psi^a_2}{\partial x^3}\right) +_n \frac{\partial \psi^a_2}{\partial x^0} = 0.\overbrace{\underbrace{0\ldots0}_{n-35}*\ldots*}^{n}$$

$$\left(\left(\frac{\partial \psi^b_1}{\partial x^1} +_n \frac{\partial \psi^a_1}{\partial x^2}\right) -_n \frac{\partial \psi^b_2}{\partial x^3}\right) +_n \frac{\partial \psi^b_2}{\partial x^0} = 0.\overbrace{\underbrace{0\ldots0}_{n-35}*\ldots*}^{n}$$

where any $* \in \{0, 1, \ldots, 9\}$ and is random.

THEOREM 8.2. *Let $n > 35$, $0 < k < n$, and $m_0 \times_n c = 0.\overbrace{\underbrace{0\ldots0}_{k}*\ldots*}^{n}$, also let*

$$\left(\left(\frac{\partial \psi^a_{\hat{1}}}{\partial x^3} +_n \frac{\partial \psi^a_{\hat{1}}}{\partial x^0}\right) +_n \frac{\partial \psi^a_{\hat{2}}}{\partial x^1}\right) -_n \frac{\partial \psi^b_{\hat{2}}}{\partial x^2} = 0.\overbrace{\underbrace{0\ldots0}_{n-35}*\ldots*}^{n}$$

$$\left(\left(\frac{\partial \psi^b_{\hat{1}}}{\partial x^3} +_n \frac{\partial \psi^b_{\hat{1}}}{\partial x^0}\right) +_n \frac{\partial \psi^b_{\hat{2}}}{\partial x^1}\right) +_n \frac{\partial \psi^b_{\hat{2}}}{\partial x^3} = 0.\overbrace{\underbrace{0\ldots0}_{n-35}*\ldots*}^{n}$$

$$\left(\left(\frac{\partial\psi^a_{\hat{1}}}{\partial x^1} +_n \frac{\partial\psi^b_{\hat{1}}}{\partial x^2}\right) -_n \frac{\partial\psi^a_{\hat{2}}}{\partial x^3}\right) +_n \frac{\partial\psi^b_{\hat{2}}}{\partial x^0} = 0.\overbrace{\underbrace{0\ldots0}_{n-35}*\ldots*}^{n}$$

$$\left(\left(\frac{\partial\psi^b_{\hat{1}}}{\partial x^1} -_n \frac{\partial\psi^b_{\hat{1}}}{\partial x^2}\right) -_n \frac{\partial\psi^b_{\hat{2}}}{\partial x^3}\right) +_n \frac{\partial\psi^b_{\hat{2}}}{\partial x^0} = 0.\overbrace{\underbrace{0\ldots0}_{n-35}*\ldots*}^{n}$$

$$\left(\left(\frac{\partial\psi^a_{1}}{\partial x^3} +_n \frac{\partial\psi^a_{1}}{\partial x^0}\right) +_n \frac{\partial\psi^a_{2}}{\partial x^1}\right) +_n \frac{\partial\psi^b_{2}}{\partial x^2} = 0.\overbrace{\underbrace{0\ldots0}_{n-35}*\ldots*}^{n}$$

$$\left(\left(\frac{\partial\psi^b_{1}}{\partial x^3} +_n \frac{\partial\psi^b_{1}}{\partial x^0}\right) +_n \frac{\partial\psi^b_{2}}{\partial x^1}\right) -_n \frac{\partial\psi^b_{2}}{\partial x^2} = 0.\overbrace{\underbrace{0\ldots0}_{n-35}*\ldots*}^{n}$$

$$\left(\left(\frac{\partial\psi^a_{1}}{\partial x^1} -_n \frac{\partial\psi^b_{1}}{\partial x^2}\right) -_n \frac{\partial\psi^a_{2}}{\partial x^3}\right) +_n \frac{\partial\psi^a_{2}}{\partial x^0} = 0.\overbrace{\underbrace{0\ldots0}_{n-35}*\ldots*}^{n}$$

$$\left(\left(\frac{\partial\psi^b_{1}}{\partial x^1} +_n \frac{\partial\psi^a_{1}}{\partial x^2}\right) -_n \frac{\partial\psi^b_{2}}{\partial x^3}\right) +_n \frac{\partial\psi^b_{2}}{\partial x^0} = 0.\overbrace{\underbrace{0\ldots0}_{n-35}*\ldots*}^{n}$$

where any $* \in \{0, 1, \ldots, 9\}$, *then*

$$\psi^a_1 = 0.\overbrace{\underbrace{0\ldots0}_{n-k}*\ldots*}^{n},$$

$$\psi^b_1 = 0.\overbrace{\underbrace{0\ldots0}_{n-k}*\ldots*}^{n},$$

$$\psi^a_2 = 0.\overbrace{\underbrace{0\ldots0}_{n-k}*\ldots*}^{n},$$

$$\psi^b_2 = 0.\overbrace{\underbrace{0\ldots0}_{n-k}*\ldots*}^{n},$$

$$\psi^a_{\hat{1}} = 0.\overbrace{\underbrace{0\ldots0}_{n-k}*\ldots*}^{n},$$

$$\psi^b_{\hat{1}} = 0.\overbrace{\underbrace{0\ldots0}_{n-k}*\ldots*}^{n},$$

$$\psi^a_{\hat{2}} = 0.\overbrace{\underbrace{0\ldots0}_{n-k}*\ldots*}^{n},$$

$$\psi^b_{\hat{2}} = 0.\overbrace{\underbrace{0\ldots0}_{n-k}*\ldots*}^{n}$$

where any $* \in \{0, 1, \ldots, 9\}$ *and is random. Thus,* ψ^a_1, ψ^b_1,ψ^a_2, ψ^b_2, $\psi^a_{\hat{1}}$, $\psi^b_{\hat{1}}$, $\psi^a_{\hat{2}}$, *and* $\psi^b_{\hat{2}}$ *are random variables.*

Proofs of both of the above theorems follow from the arithmetic rules.

9. SOLITARY WAVES AND DISPERSIVE EQUATIONS FROM OBSERVERS MATHEMATICS POINT OF VIEW

In classical physics, it has been realized for centuries that the behavior of idealized vibrating media (such as waves on string, on a water surface, or in air), in the absence of friction or other dissipative forces, can be modeled by a number of partial differential equations, known collectively as dispersive equations. Model examples of such equations include the following:

- The free wave equation

  $$u_{tt} - c^2 \Delta u = 0$$

  where $u : R \times R^d \to R$ represents the amplitude $u(t, x)$ of a wave at a point in spacetime with d spatial dimensions, $\Delta = \sum_{j=1}^{d} \frac{\delta^2}{\delta x_j^2}$ is the spatial Laplacian on R^d, u_{tt} is short for $\frac{\delta^2 u}{\delta t^2}$, and $c > 0$ is a fixed constant.

- The linear Schrodinger equation

  $$i\hbar u_t + \frac{\hbar^2}{2m} \Delta u = Vu$$

  where $u : R \times R^d \to R$ is the wave function of a quantum particle, $\hbar, m > 0$ are physical constants and $V : R^d \to R$ is a potential function, which we assume to depend only on the spatial variable x.

- The Airy equation

  $$u_t + u_{xxx} = 0$$

  where $u : R \times R \to R$ is a scalar function.

- The Korteweg-de Vries equation

$$u_t + u_{xxx} + 6uu_x = 0$$

  which is a more refined version of the Airy equation in which the first nonlinear term is retained.

The theory of linear dispersive equations predicts that waves should spread out and disperse over time. However, it is a remarkable phenomenon, observed both in theory and practice, that once nonlinear effects are taken into account, solitary wave and soliton solutions can be created, which can be stable enough to persist indefinitely. In this section we consider some properties of these equations from Observer's Mathematics point of view, see 30, 31, 32, 33, and 34.

9.1 Free Wave Equation

We consider the case when $d = 1$, i.e., $u : W_n \times W_n \to W_n$, from W_m-observer point of view, with $m > n$, where $W_n \times W_n$ means Cartesian product of W_n with itself. The free wave equation may be written as

$$u_{tt} -_n ((c \times_n c) \times_n u_{xx}) = 0$$

Then we have the following

THEOREM 9.1. *Let*

$$c = c_0.c_1 \ldots c_k c_{k+1} \ldots c_n$$

and

$$u_{xx} = \pm u_0^{xx}.u_1^{xx} \ldots u_l^{xx} u_{l+1}^{xx} \ldots u_n^{xx}$$

with $2k < n$, $l < n$, $c_0 = c_1 = \ldots = c_k = 0$, $c_{k+1} \neq 0$, $u_0^{xx} = u_1^{xx} = \ldots = u_l^{xx} = 0$ *and* $u < k + l + 2$, *then* $u_{tt} = 0$.

Proof. We have $c \times_n c = d_0.d_1 \ldots d_r d_{r+1} \ldots d_n$, with $d_0 = d_1 = \ldots = d_r = 0$, if $r = 2k$, $d_{r+1} \neq 0$. Thus, $(c \times_n c) \times_n u_{xx} = 0$ and $u_{tt} = 0$. □

Next, we have the following

THEOREM 9.2. *If* $d_0 \geq \underbrace{9 \ldots 9}_{p}$, *with* $0 < p \leq n$ *and* $u_0^{xx} \geq \underbrace{9 \ldots 9}_{q}$, *with* $0 < q \leq n$ *and* $n < p + q$, *then there is no* u_{tt}, *such that* $u_{tt} = ((c \times_n c) \times_n u_{xx})$.

Proof. $(c \times_n c) \times_n |u_{xx}| > d_0 \times_n u_0^{xx}$, but $d_0 \times_n u_0^{xx}$ does not exist in W_n. □

9.2 Schrodinger Equation

Consider the following: $-(\hbar \times_n \hbar) \times_n \Psi_{xx} +_n ((2 \times_n m) \times_n V) \times_n \Psi = i((2 \times_n m) \times_n \hbar)\Psi_t$, where $\Psi = \Psi(x,t)$, $\hbar$ is the Planck's Constant, $\hbar = 1.054571628(53) \times 10^{-34}\ m^2kg/s$. $\Psi \in CW_n$, $\Psi = \Psi^a + i\Psi^b$. In the following statements we speak about $\Psi^a_x x$, $\Psi^b_x x$, Ψ^a_t, Ψ^b_t, Ψ^a, and Ψ^b.

Then we have the following

THEOREM 9.3. *Let* $36 < n < 68$, $m = m_0.m_1 \ldots m_k m_{k+1} \ldots m_n$, *with* $m \in W_n$, $m_0 = m_1 = \ldots = m_k = 0$, $m_{k+1} \neq 0$, $k + 35 < n$, $V = 0$, *then* $\Psi_t = \Psi^0_t.\Psi^1_t \ldots \Psi^l_t \Psi^{l+1}_t \ldots \Psi^n_t$ *and* $\Psi^0_t = \ldots \Psi^l_t = 0$, $\Psi^{l+1}_t, \ldots, \Psi^n_t$ *are free and in* $\{0, 1, \ldots, 9\}$, *where* $l = n - k - 36$, *i.e.,* Ψ_t *is a random variable, with* $\Psi_t \in \{(0.\overbrace{\underbrace{0 \ldots 0}_{l} * \ldots *}^{n})\}$, *where* $* \in \{0, 1, \ldots, 9\}$.

Proof. We have $-(\hbar \times_n \hbar) = 0$, $(2 \times_n m) \times_n V = 0$, i.e., $i((2 \times_n m) \times_n \hbar)\Psi_t = 0$ and $l = n - k - 36$. □

COROLLARY 9.4. *Let* $36 < n < 68$, $m = m_0.m_1 \ldots m_k m_{k+1} \ldots m_n$, *with* $m \in W_n$, $m_0 = m_1 = \ldots = m_k = 0$, $m_{k+1} \neq 0$. *Also, let* $V = \upsilon_0.\upsilon_1 \ldots \upsilon_s \upsilon_{s+1} \ldots \upsilon_n$, *with* $V \in W_n$, $\upsilon_0 = \upsilon_1 = \ldots = \upsilon_s = 0$, $\upsilon_{s+1} \neq 0$, *with* $\begin{cases} k + 35 & < & n \\ k + s + 2 & > & n \end{cases}$, *then* $\Psi_t = \Psi^0_t.\Psi^1_t \ldots \Psi^l_t \Psi^{l+1}_t \ldots \Psi^n_t$ *and* $\Psi^0_t = \ldots \Psi^l_t = 0$, $\Psi^{l+1}_t, \ldots, \Psi^n_t$ *are free and in* $\{0, 1, \ldots, 9\}$, *where* $l = n - k - 36$, *i.e.,* Ψ_t *is a random variable, with* $\Psi_t \in \{(0.\overbrace{\underbrace{0 \ldots 0}_{l} * \ldots *}^{n})\}$, *where* $* \in \{0, 1, \ldots, 9\}$.

9.3 Two-Slit Interference

Quantum mechanics treats the motion of an electron, neutron or atom by writing down the Schrodinger equation:

$$-\frac{\hbar^2}{2m}\frac{\delta^2\Psi}{\delta x^2} + V\Psi = i\hbar\frac{\delta\Psi}{\delta t}$$

where m is the particle mass and V is the external potential acting on the particle. As these particles pass through the two slits of any of the experiments they are moving freely; we, therefore, set $V = 0$ in the Schrodinger equation.

Now, consider the following:

$$-(\hbar \times_n \hbar) \times_n \Psi_{xx} +_n ((2 \times_n m) \times_n V) \times_n \Psi = i((2 \times_n m) \times_n \hbar)\Psi_t$$

where $\Psi = \Psi(x,t)$, $\hbar$ is the Planck's Constant, $\hbar = 1.054571628(53) \times 10^{-34}\ m^2kg/s$. Then we have the following

THEOREM 9.5. *Let* $36 < n < 68$, $m = m_0.m_1 \ldots m_k m_{k+1} \ldots m_n$, *with* $m \in W_n$, $m_0 = m_1 = \ldots = m_k = 0$, $m_{k+1} \neq 0$, $k + 35 < n$, $V = 0$, *then* $\Psi_t = \Psi_t^0.\Psi_t^1 \ldots \Psi_t^l \Psi_t^{l+1} \ldots \Psi_t^n$ *and* $\Psi_t^0 = \ldots \Psi_t^l = 0$, $\Psi_t^{l+1}, \ldots, \Psi_t^n$ *are free and in* $\{0, 1, \ldots, 9\}$, *where* $l = n - k - 36$, *i.e.,* Ψ_t *is a random variable, with* $\Psi_t \in \{(0.\overbrace{\underbrace{0\ldots0}_{l}*\ldots*}^{n})\}$, *where* $* \in \{0, 1, \ldots, 9\}$.

The wave at the point of combination will be the sum of those from each slit. If Ψ_1 is the wave from slit 1 and Ψ_2 is the wave from slit 2, then $\Psi = \Psi_1 + \Psi_2$. The result gives the predicted interference pattern. Then we have

$$\Psi_{1t} = \Psi_{1t}^0.\Psi_{1t}^1 \ldots \Psi_{1t}^l \Psi_{1t}^{l+1} \ldots \Psi_{1t}^n$$

$$\Psi_{2t} = \Psi_{2t}^0.\Psi_{2t}^1 \ldots \Psi_{2t}^l \Psi_{2t}^{l+1} \ldots \Psi_{2t}^n$$

$$\Psi_{1t}^0 = \ldots = \Psi_{1t}^l = 0$$

Where $\Psi_{1t}^{l_1+1}, \ldots, \Psi_{1t}^n$ are free and in $\{0, 1, \ldots, 9\}$. and

$$\Psi_{2t}^0 = \ldots = \Psi_{2t}^l = 0$$

Where $\Psi_{2t}^{l_2+1}, \ldots, \Psi_{2t}^n$ are free and in $\{0, 1, \ldots, 9\}$ where $l = n - k - 36$.

Now we have the following

Theorem 9.6.

1. *If* $\Psi_{1t}^{l+1} + \Psi_{2t}^{l+1} > 9$, *then* $\Psi_1 + \Psi_2$ *is not a wave.*

2. *If* $\Psi_{1t}^{l+1} + \Psi_{2t}^{l+1} < 9$, *then* $\Psi_1 + \Psi_2$ *is a wave.*

3. *If* $\Psi_{1t}^{l+1} + \Psi_{2t}^{l+1} = 9$, *then* $\Psi_1 + \Psi_2$ *may or may not be a wave.*

Proof. If $\Psi = \Psi_1 + \Psi_2$ is a wave, we have to have $\Psi = \Psi_0.\Psi_1 \ldots \Psi_l \Psi_{l+1} \ldots \Psi_n$ (which is necessary and sufficient) with $\Psi_0 = \Psi_1 = \ldots = \Psi_l = 0$. Thus, for the first statement in the theorem, we have $\Psi_l \neq 0$ and for the second statement, we must have $\Psi_l = 0$. For the third case both variants are possible. □

9.4 Airy and Korteweg-de Vries Equations

If $u : W_n \times W_n \to W_n$ then the Airy equation may be written as

$$u_t +_n u_{xxx} = 0$$

and Korteweg-de Vries equation may be written as

$$(u_t +_n u_{xxx}) +_n 6(u \times_n u_x) = 0$$

Then we have the following

THEOREM 9.7. *Let*

$$u = u_0.u_1 \ldots u_k u_{k+1} \ldots u_n$$

and

$$u_x = u_0^x.u_1^x \ldots u_l^x u_{l+1}^x \ldots u_n^x$$

with $k < n$, $l \leq n$ *and* $u_0 = u_1 = \ldots = u_k = 0$ *and* $u_0^x = u_1^x = \ldots = u_l^x = 0$ *and* $k + l > n$, *then Airy equation and Korteweg-de Vries equation have the solution.*

Proof. In this case $u \times_n u_x = 0$ and $u_t +_n u_{xxx} = 0$. □

9.5 Schwartzian Derivative

The Schwartzian derivative $S(f(x))$ is defined as

$$S(f(x)) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

Here $f(x)$ is a function in one real variable and $f'(x), f''(x), f'''(x)$ are its derivatives. The Schwartzian derivative is ubiquitous and tends to appear in seemingly unrelated fields of Mathematics including classical complex analysis, differential equations, and one-dimensional analysis, as well as more recently, Teichmüller Theory, integrable systems, and conformal field theory. For example, let's consider the Lorentz plane with the metric $g = dxdy$ and a curve $y = f(x)$. If $f'(x) > 0$, then its Lorentz curvature can be easily computed via

$$\rho(x) = f''(x)(f'(x))^{-\frac{3}{2}}$$

and the Schwartzian enters the game when one computes $\rho' = \frac{S(f)}{\sqrt{f'}}$. Thus, informally speaking, the Schwartzian derivative is curvature.

Consider now the Schwartzian curvature from Observer's Mathematics point of view.

Now we have the following

THEOREM 9.8. *If $S(f(x))$ exists, then*

- *$S(f(x))$ is a random variable.*
- *$|S(f(x)| \leq 10^{l-k+1}$, where*

$$(2 \times_n (f'(x) \times_n f'(x))) = 0.\underbrace{0\ldots0a_l}_{l}a_{l+1}\ldots a_n$$

with $a_l \neq 0$ *and*

$$(2 \times_n (f'''(x) \times_n f'(x))) -_n (3 \times_n (f''(x) \times_n f''(x))) = \pm 0.\underbrace{0\ldots0b_k}_{k} b_{k+1}\ldots b_n$$

with $b_k \neq 0$ *and* $1 < l, k < n$.

Proof. We have

$$\pm 0.\underbrace{0\ldots0a_l}_{l} a_{l+1}\ldots a_n \times_n S(f(x)) = \pm 0.\underbrace{0\ldots0b_k}_{k} b_{k+1}\ldots b_n$$

If $S(f(x)) = c_0.c_1\ldots c_n$ then $c_m, c_{m+1}, \ldots, c_n$ are free and in $\{0,\ldots,9\}$ with $m + l > n$. Then we have:

- If $l > k$, then $|S(f(x))| \leq 10^{l-k}$.
- If $l = k$, then $|S(f(x))| \leq 10$.
- If $l < k$, then $|S(f(x))| \leq 10^{l-k+1}$.

□

9.6 Newton equation

Let $F(x,t) = m \times_n \ddot{x}$. Then we have the following

THEOREM 9.9. *If the body with mass* $m = m_0.m_1 \ldots m_k m_{k+1} \ldots m_n$*, with* $m \in W_n$*, moves with acceleration* $\ddot{x}$*,* $|\ddot{x}| = \ddot{x}_0.\ddot{x}_1 \ldots \ddot{x}_l \ddot{x}_{l+1} \ldots \ddot{x}_n$*, with* $\ddot{x} \in W_n$*, and* $m_0 = m_1 = \ldots = m_k = 0$*,* $m_{k+1} \neq 0$*,* $k < n$*,* $\ddot{x}_0 = \ddot{x}_1 = \ldots = \ddot{x}_l = 0$*,* $l < n$*,* $k + l + 2 \in W_n$*,* $n < k + l + 2 \leq q$*, then* $F(x,t) = 0$*.*

Proof. In this case $m \times_n |\ddot{x}| = 0$. □

COROLLARY 9.10. *If* $l = n - 1$ *and* $k = 0$*, i.e.,* $m < 1$*, then* $F(x,t)$*.*

THEOREM 9.11. *If* $l = n - 1$ *and* $\ddot{x}_n \neq 0$ *then* $|F(x,t)| < 9$*.*

Proof. $|F(x,t)| = m \times_n |\ddot{x}| \leq \underbrace{9 \ldots 9}_{n}.\underbrace{9 \ldots 9}_{n} \times_n 0.\underbrace{0 \ldots 0}_{n-1}9 = 8.\underbrace{9 \ldots 9}_{n-1}1 < 9$. □

THEOREM 9.12. *If* $m_0 \geq \underbrace{9 \ldots 9}_{p}$*,* $0 < p \leq n$*,* $\ddot{x}_0 \geq \underbrace{9 \ldots 9}_{r}$*,* $0 < r \leq n$*,* $n < p + r \leq q$*, then there is no force* $F(x,t)$*, such that* $F(x,t) = m \times_n \ddot{x}$*.*

Proof. $m \times_n \ddot{x} > m_0 \times_n \ddot{x}_0$ does not exist in W_n. □

9.7 Geodesic equation

Consider the following:

$$\ddot{x}^i +_n \sum_j {}^n \sum_k {}^n\Gamma^i_{jk} \times_n (\dot{x}^j \times_n \dot{x}^k) = 0$$

with $j, k \in G$. Then we have the following

THEOREM 9.13. *If* $\dot{x}^p = \dot{x}^p_0.\dot{x}^p_1 \ldots \dot{x}^p_l \dot{x}^p_{l+1} \ldots \dot{x}^p_n$, *with* $p \in G$, $\dot{x}^p_0 = \dot{x}^p_1 = \ldots = \dot{x}^p_l = 0$, $0 \leq l \leq n$, $n < 2l \leq q$, *then we have* $\ddot{x}^i = 0$, *i.e., the geodesic curve is a line.*

Proof. $\dot{x}^j \times_n \dot{x}^k = \dot{x}^j_0.\dot{x}^j_1 \ldots \dot{x}^j_l \dot{x}^j_{l+1} \ldots \dot{x}^j_n \times_n \dot{x}^k_0.\dot{x}^k_1 \ldots \dot{x}^k_l \dot{x}^k_{l+1} \ldots \dot{x}^k_n = (0.0 \ldots 0\dot{x}^j_{l+1} \ldots \dot{x}^j_n \times_n 0.0 \ldots 0\dot{x}^k_{l+1} \ldots \dot{x}^k_n) = 0$. ∎

9.8 Wave-Particle Duality for Single Photons

The connection between interference as a characteristic of waves and particles was noticed by de Broglie. He connected particle to wave mechanics. He proposed that particles behave as if they possessed a wavelength that was inversely proportional to their momentum, mV, and that the constant of proportionality was Planck's constant $\hbar$:

$$\lambda = \frac{\hbar}{mV}$$

I.e. we have

$$\lambda mV = \hbar$$

We have the following equation in Observer's Mathematics: $\lambda \times_n (m \times_n V) = \hbar$ with $\hbar = 1.\ldots \times 10^{-34}$ for $n > 60$. And now we have the following

THEOREM 9.14. *If* $m \times_n V = c_0.c_1 \ldots c_k c_{k+1} \ldots c_n$ *with* $c_0 = c_1 = \ldots = c_k = 0$ *and* $c_{k+1} \neq 0$, *with* $k < n$ *and* $\lambda = \lambda_0.\lambda_1 \ldots \lambda_m \lambda_{m+1} \ldots \lambda_n$ *with* $m < n$ *and* $m + k > n$, *then* $\lambda_{m+1}, \ldots, \lambda_n$ *are free and in* $\{0, 1, \ldots, 9\}$ *and* $\lambda_0.\lambda_1 \ldots \lambda_m \times_n 0.\underbrace{0\ldots0}_{k} c_{k+1} \ldots c_n = \hbar$.

Proof. Given that $m + k > n$, we have

$$0.\underbrace{0\ldots0}_{m}\lambda_{m+1}\ldots\lambda_n \cdot 0.\underbrace{0\ldots0}_{k}c_{k+1}\ldots c_n = 0$$

□

9.9 Uncertainty Principle

Using the de Broglie relationship between momentum and wave number, $p = (\hbar)k$, we can obtain the position-momentum uncertainty relationship:

$$\Delta p \cdot \Delta x = \hbar$$

THEOREM 9.15. *Let* $\Delta p = p_0.p_1 \dots p_k p_{k+1} \dots p_n$ *with* $k < n$ *and* $\Delta x = x_0.x_1 \dots x_l x_{l+1} \dots x_n$ *with* $l < n$. *Then*

1. *If* $p_0 = p_1 = \dots = p_k = 0$ *and* $k + l > n$, *then* $x_{l+1}, \dots, x_n$ *are free and in* $\{0, 1, \dots, 9\}$.

2. *If* $x_0 = x_1 = \dots = x_l = 0$ *and* $k + l > n$, *then* $p_{k+1}, \dots, p_n$ *are free and in* $\{0, 1, \dots, 9\}$.

3. *If* $p_0 = p_1 = \dots = p_k = 0$ *and* $x_0 = x_1 = \dots = x_l = 0$ *then* $k + l \leq 34$.

Proof. The proof is derived using arithmetic defined in Observer's Mathematics. □

10. SPECIAL RELATIVITY FROM OBSERVERS MATHEMATICS POINT OF VIEW

10.1 Classical Lorentz transformation

The following discussion is based 35. For the relative orientation of the coordinate systems K and K', the $x-$axes of both systems permanently coincide. In the present case we consider only events which are localized on the x-axis. Any such event is represented with respect to the coordinate system K by the abscissa x and the time t, and with respect to the system K' by the abscissa x' and the time t', when x and t are given. We call v the velocity with which the origin of K' is moving relative to K. A light-signal, which is proceeding along the positive axis of x, is transmitted according to the equation

$$x = ct$$

or

$$x - ct = 0$$

Since the same light-signal has to be transmitted relative to K' with the velocity c, the propagation relative to the system K' will be represented by the analogous formula $x' - ct' = 0$. At that the disappearance of $(x - ct)$ involves the disappearance of $(x' - ct')$, and vice versa. If we apply quite similar considerations to light rays which are being transmitted along the negative $x-$axis, we obtain the analogous condition: the disappearance of $(x + ct)$ involves the disappearance of $(x' + ct')$, and vice versa.

Classical way to use this first principal of Special Theory of Relativity is the following. We would like to say that this will be the case when the relation

$$x' - ct' = \lambda(x - ct) \tag{1}$$

is fulfilled in general, where λ indicates a constant, $\lambda \neq 0$. Also, we would like to say that this will be the case when the relation

$$x' + ct' = \mu(x + ct) \tag{2}$$

is fulfilled in general, where μ indicates a constant, $\mu \neq 0$. So, we have to write down the first principal of Special Theory of Relativity using the following equalities:

$$\begin{cases} x' - ct' = \lambda(x - ct) \\ x' + ct' = \mu(x + ct) \end{cases} \tag{3}$$

where λ, μ are constants, $\lambda, \mu \neq 0$. For the origin of K' we have permanently $x' = 0$, and $x = vt$, where v is the velocity with which the origin of K' is moving relative to K, $0 < v < c$. It means:

$$\begin{cases} -ct' = \lambda(vt - ct) \\ ct' = \mu(vt + ct) \end{cases} \tag{4}$$

From here we have

$$-\lambda(vt - ct) = \mu(vt + ct)$$

or

$$\lambda(c - v) = \mu(c + v) \tag{5}$$

Furthermore, the second principle of relativity states that, as judged from K, the length of a unit measuring-rod which is at rest with reference to K' must be exactly the same as the length, as judged from K', of a unit measuring-rod which is at rest relative to K. In order to see how the points of the x'-axis appear as viewed from K, we only require to take a "snapshot" of K' from K; this means that we have to insert a particular value of t (time of K), e.g. $t = 0$. For this value of t we then obtain from the first set of the equations

$$\begin{cases} x' - ct' = \lambda x \\ x' + ct' = \mu x \end{cases} \tag{6}$$

So,

$$ct' = x' - \lambda x$$

and

$$2x' - \lambda x = \mu x$$

Let's take $x'_0 = 0$, $x'_1 = 1$, then find corresponding x_0 and x_1.

$$x_0 = 0 \tag{7}$$

$$x_1 = \frac{2}{\lambda + \mu} \tag{8}$$

But if the snapshot is to be taken from K' ($t' = 0$), we obtain from the second set of the equations

$$\begin{cases} x' = \lambda(x - ct) \\ x' = \mu(x + ct) \end{cases} \tag{9}$$

$$\begin{cases} ct = \frac{(\lambda x - x')}{\lambda} \\ x' = \mu x + \frac{\mu(\lambda x - x')}{\lambda} \end{cases} \tag{10}$$

Let's take $x_3 = 0$, $x_4 = 1$, then find corresponding x'_3 and x'_4.

$$x'_3 = 0 \tag{11}$$

$$\lambda x_4' = \lambda\mu + \lambda\mu - \mu x_4'$$

$$x_4' = \frac{2\lambda\mu}{\lambda + \mu} \tag{12}$$

But from what has been said, the two snapshots must be identical; hence x_1 must be equal to x_1', so that we obtain:

$$x_1 = x_4'$$

That means

$$\frac{2}{(\lambda + \mu)} = \frac{2\lambda\mu}{(\lambda + \mu)}$$

i.e.,

$$\lambda\mu = 1$$

So, we have a system

$$\begin{cases} \lambda(c - v) = \mu(c + v) \\ \lambda\mu = 1 \end{cases} \tag{13}$$

Solution of this system is

$$\begin{cases} \lambda = \sqrt{\frac{c+v}{c-v}} \\ \mu = \sqrt{\frac{c-v}{c+v}} \end{cases} \tag{14}$$

Standard expression of Lorentz transformation we get if we put these values of λ and μ, and solve the system of equations 3. We get:

$$\begin{cases} x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \\ t' = \frac{\frac{v}{c^2}x + t}{\sqrt{1 - \frac{v^2}{c^2}}} \end{cases} \tag{15}$$

10.2 Zero-divisors, non-associativity and non-distributivity, Lorentz transformation in Observer's Mathematics

Let us consider the Observer's Mathematics point of view, see 5. We consider all events below as appurtenant to W_n for some n, and point of view belongs to W_m with $m > n$. Here we do not take numerical estimation of m, but for us it is enough that W_m observer can see all sets of numbers which we operate on each step. A light-signal, which is proceeding along the positive axis of x, is transmitted according to the equation

$$x = c \times_n t$$

or

$$x -_n c \times_n t = 0$$

Since the same light-signal has to be transmitted relative to K' with the velocity c, the propagation relative to the system K' will be represented by the analogous formula $x' -_n c \times_n t' = 0$. At that the disappearance of $(x -_n c \times_n t)$ involves the disappearance of $(x' -_n c \times_n t')$, and vice versa. If we apply quite similar considerations to light rays which are being transmitted along the negative x-axis, we obtain the analogous condition:

$$x +_n c \times_n t = 0$$

and

$$x' +_n c \times_n t' = 0$$

And also at that the disappearance of $(x +_n c \times_n t)$ involves the disappearance of $(x' +_n c \times_n t')$, and vice versa. We would like to say that this will be the case when the relation

$$x' -_n c \times_n t' = \lambda \times_n (x -_n c \times_n t)$$

is fulfilled in general, where λ indicates a constant, $\lambda \neq 0$. Also we would like to say that this will be the case when the relation

$$x' +_n c \times_n t' = \mu \times_n (x +_n c \times_n t)$$

is fulfilled in general, where μ indicates a constant, $\mu \neq 0$.

The critical aspect of this paper is that all of these statements are wrong in Observer's Mathematics, because Observer's Mathematics has zero-divisors, see 1 and 2. For example, if we take $n = 2$, $\lambda = 0.8$ and $x -_n c \times_n t = 0.08$ then $x' -_n c \times_n t' = \lambda \times_n (x -_n c \times_n t) = 0$. Same situation takes a place with μ. Thus, if we have $|\lambda| < 1$, then the statement "the case when the relation $x' -_n c \times_n t' = \lambda \times_n (x -_n c \times_n t)$ is fulfilled in general" becomes wrong. And also if we have $|\mu| < 1$, statement "the case when the relation $x' +_n c \times_n t' = \mu \times_n (x +_n c \times_n t)$ is fulfilled in general" becomes wrong. So, relations 1 and 2 above become wrong from Observer's Mathematics point of view. But in case $\lambda \geq 1, \mu \geq 1$ both statements are correct, see see 1 and 2. We proved above that in classical case we have relation $\lambda\mu = 1$ (it is classical multiplication here). It means that if $\lambda > 1$ (and in reality $\lambda > 1$), then $\mu < 1$. We can see analogous situation in Observer's Mathematics case (see proof in chapter 4 of this paper). So, we have to change classical approach and write down the first principal of Special Theory of Relativity using the following equalities:

$$\begin{cases} x' -_n c \times_n t' = \lambda \times_n (x -_n c \times_n t) \\ \mu \times_n (x' +_n c \times_n t') = x +_n c \times_n t \end{cases} \tag{16}$$

where $\lambda \geq 1$, $\mu \geq 1$, and λ, μ are constants. For the origin of K' we have permanently $x' = 0$, and $x = v \times_n t$. It means:

$$\begin{cases} -c \times_n t' = \lambda \times_n (v \times_n t -_n c \times_n t) \\ \mu \times_n (c \times_n t') = v \times_n t +_n c \times_n t \end{cases} \tag{17}$$

From here we have

$$\mu \times_n (-\lambda \times_n (v \times_n t -_n c \times_n t)) = v \times_n t +_n c \times_n t$$

for all t. Before going forward we are going to Observer's Mathematics arithmetic. We have proved theorems 2.5 and 2.6 above. For reference, the theorems are restated as the following two theorems:

THEOREM 10.1. *Let* $\delta_2 = c \times_n (a +_n b) -_n (c \times_n a +_n c \times_n b)$. *Then* $0 < P(\delta_2 = 0) < 1$, *where* P *is the probability.*

THEOREM 10.2. *Let* $\delta_3 = c \times_n (a \times_n b) -_n (c \times_n a) \times_n b$. *Then* $0 < P(\delta_3 = 0) < 1$, *where* P *is the probability.*

Now we are going back to the equation:

$$\mu \times_n (\lambda \times_n (c \times_n t -_n v \times_n t)) = c \times_n t +_n v \times_n t$$

for all t. With probability P_1, $0 < P_1 < 1$ (by Theorem 10.1) we can rewrite this equation as:

$$\mu \times_n (\lambda \times_n ((c -_n v) \times_n t)) = (c +_n v) \times_n t$$

With probability P_2, $0 < P_2 < 1$ (by Theorem 10.2) we can rewrite this equation as:

$$\mu \times_n ((\lambda \times_n (c -_n v)) \times_n t) = (c +_n v) \times_n t$$

With probability P_3, $0 < P_3 < 1$ (by Theorem 10.2) we can rewrite this equation as:

$$(\mu \times_n (\lambda \times_n (c -_n v)) \times_n t = (c +_n v) \times_n t$$

With probability P_4, $0 < P_4 < 1$ (by Theorem 10.1) we can rewrite this equation as:

$$((\mu \times_n (\lambda \times_n (c -_n v)) -_n (c +_n v)) \times_n t = 0$$

Because this equation has to be fulfilled for all t, we can rewrite this equation as:

$$\mu \times_n (\lambda \times_n (c -_n v)) = c +_n v$$

THEOREM 10.3. *The statement: "Equation*

$$\mu \times_n (\lambda \times_n (c -_n v)) = c +_n v$$

is equivalent to equation

$$\mu \times_n (\lambda \times_n (c \times_n t -_n v \times_n t)) = c \times_n t +_n v \times_n t$$

for all t" has probability more than 0 *and less than* 1.

THEOREM 10.4. *The statement "Equation* $(\mu \times_n \lambda) \times_n (c -_n v) = c +_n v$ *is equivalent to equation* $\mu \times_n (\lambda \times_n (c \times_n t -_n v \times_n t)) = c \times_n t +_n v \times_n t$ *for all* t*" has probability more than* 0 *and less than* 1.

Note, if we are dealing with a space-time with $|t| < 1$, then Theorem 10.3 becomes incorrect because there are zero-divisors in W_n. So, relation 5 above may be correct only with probability which is more than 0 but less than 1, from Observer's Mathematics point of view. Furthermore, the second principle of relativity states that, as judged from K, the length of a unit measuring-rod which is at rest with reference to K' must be exactly the same as the length, as judged from K', of a unit measuring-rod which is at rest relative to K. In order to see how the points of the x'-axis appear as viewed from K, we only require to take a "snapshot" of K' from K; this means that we have to insert a particular value of t (time of K), e.g. $t = 0$. For this value of t we then obtain from the first set of the equations 16. So, $c \times_n t' = x' -_n \lambda \times_n x$ and $\mu \times_n (2 \times_n x' -_n \lambda \times_n x) = x$. Let's take $x'_0 = 0$ and $x'_1 = 1$ and then find corresponding x_0 and x_1. Then we get the following:

$$\mu \times_n (-\lambda \times_n x_0) = x_0$$

With probability P_5 , $0 < P_5 < 1$ (by Theorem 10.2), we can rewrite this equation as:

$$(\mu \times_n \lambda) \times_n x_0 +_n x_0 = 0$$

With probability P_6 , $0 < P_6 < 1$ (by Theorem 10.1), we can rewrite this equation as:

$$((\mu \times_n \lambda) +_n 1) \times_n x_0 = 0$$

If $\mu \times_n \lambda > 0$, $x_0 = 0$. So, we proved that $x_0 = 0$ with probability P_7, $0 < P_7 < 1$. x_1 is a solution of equation:

$$\mu \times_n (2 -_n \lambda \times_n x_1) = x_1$$

But if the snapshot would be taken from K' ($t' = 0$), we obtain from the second set of the equations

$$\begin{cases} x' = \lambda \times_n (x -_n c \times_n t) \\ \mu \times_n x' = x +_n c \times_n t \end{cases} \tag{18}$$

So, $c \times_n t = \mu \times_n x' -_n x$ and $x' = \lambda \times_n (2 \times_n x -_n \mu \times_n x')$ Let's take $x_3 = 0$, $x_4 = 1$, and then find corresponding x'_3 and x'_4.

$$x'_3 = \lambda \times_n (-\mu \times_n x'_3)$$

With probability P_8 , $0 < P_8 < 1$ (by Theorem 10.2), we can rewrite this equation as:

$$x'_3 +_n (\lambda \times_n \mu) \times_n x'_3 = 0$$

With probability P_9 , $0 < P_9 < 1$ (by Theorem 10.1), we can rewrite this equation as:

$$(1 +_n (\lambda \times_n \mu)) \times_n x'_3 = 0$$

If $\mu \times_n \lambda > 0$, $x'_3 = 0$. So, we proved that $x'_3 = 0$ with probability P_{10}, $0 < P_{10} < 1$. x'_4 is a solution of equation: $x'_4 = \lambda \times_n (2 -_n \mu \times_n x'_4)$ But from what has been said, the two snapshots must be identical; hence x_1 must be equal to x'_4, so that we obtain:

$$x_1 = x'_4$$

So, relations 7, 8, 11, and 12 above may be correct only with probability which is more than 0 but less than 1, from Observer's Mathematics point of view. And finally we have a system of equations:

$$\begin{cases} \mu \times_n (\lambda \times_n (c -_n v)) = c +_n v \\ \mu \times_n (2 -_n \lambda \times_n x_1) = x_1 \\ x'_4 = \lambda \times_n (2 -_n \mu \times_n x'_4) \\ x_1 = x'_4 \end{cases} \tag{19}$$

Now we denote $x_1 = x'_4 = x_2$. And we can rewrite this system as a system with 3 equations:

$$\begin{cases} \mu \times_n (\lambda \times_n (c -_n v)) = c +_n v \\ \mu \times_n (2 -_n \lambda \times_n x_2) = x_2 \\ \lambda \times_n (2 -_n \mu \times_n x_2) = x_2 \\ 0 < v < c \\ \lambda \geq 1 \\ \mu \geq 1 \end{cases} \tag{20}$$

So, general transformation given by equations 16 with λ and μ satisfying a system of equations 20 is a 2-dimensional analog of classical Lorentz transformation. We will call this transformation Observer's Mathematics Lorentz transformation. So, we proved

THEOREM 10.5. *Observer's Mathematics Lorentz transformation is satisfying to the first and second principals of Special Theory of Relativity with probability P, $0 < P < 1$.* Note, using Theorem 10.4, we can state the same if we substitute equation $\mu \times_n (\lambda \times_n (c -_n v)) = c +_n v$ for $(\mu \times_n \lambda) \times_n (c -_n v) = c +_n v$.

10.3 Observer's Mathematics Lorentz Transformation

Characteristics

Let's consider the system of equations defining constants λ and μ (with given v) of Observer's Mathematics Lorentz transformation provided by system 20.

THEOREM 10.6. *The constants λ and μ are both > 1 automatically, i.e., the system of equations defining constants λ and μ may be written downs as follows:*

$$\begin{cases} \mu \times_n (\lambda \times_n (c -_n v)) = c +_n v \\ \mu \times_n (2 -_n \lambda \times_n x_2) = x_2 \\ \lambda \times_n (2 -_n \mu \times_n x_2) = x_2 \\ 0 < v < c \end{cases} \tag{21}$$

Proof: First, let's suppose $\lambda < 1$ and $\mu < 1$. We have $c - v < c + v$. Then

$$\lambda \times_n (c - v) < c + v$$

and

$$\mu \times_n (\lambda \times_n (c - v)) < c + v$$

which is a contradiction. Let's suppose now $\lambda < 1$, $\mu > 1$, and $0 < x_2 < 1$. Then

$$\lambda \times_n x_2 < x_2 < 1$$

and

$$2 -_n \lambda \times_n x_2 > 1$$

Thus,

$$\mu \times_n (2 -_n \lambda \times_n x_2) > 1$$

and

$$\mu \times_n (2 -_n \lambda \times_n x_2) \neq x_2$$

which is another contradiction. Let's suppose now $\lambda < 1$, $\mu > 1$, and $x_2 > 1$. Then

$$\mu \times_n x_2 > 1$$

$$2 -_n \mu \times_n x_2 < 1$$

So,

$$\lambda \times_n (2 -_n \mu \times_n x_2) < 1$$

and

$$\lambda \times_n (2 -_n \mu \times_n x_2) \neq x_2$$

which is also a contradiction. The remaining possible cases (one, or two, or three λ, μ, x_2 inequalities become decent) generate the same results. Therefore, the theorem is proved.

We can now rewrite Observer's Mathematics Lorentz transformation in W_n as system 16 with λ and μ satisfying the system of equations 21.

Let's now consider solutions to the existing question. For any given v, $0 < v < c$, solutions are the sets $\{\mu, x_2\}$. Let's consider for example $n = 2$, i.e., $x, t, x', t', c, v, \lambda, \mu, x_2 \in W_2$. Put $c = 1$, then $0 < v < 1$, i.e., $v \in \{0.01, 0.02, \ldots, 0.98, 0.99\}$. Also, let's assume $\lambda = \mu$. The full set of solutions is presented in the table below (using Theorem 10.4).

v	$\lambda = \mu$	x_2
0.16	1.2; 1.21; 1.22	0.99
0.2	1.23; 1.24; 1.25; 1.26; 1.27	0.99
0.21	1.28; 1.29	0.99
	1.3	0.97
0.28	1.36; 1.37; 1.38; 1.39	0.97
0.56	1.9; 1.91; 1.92; 1.93; 1.94	0.82
0.57	1.95; 1.96; 1.97; 1.98; 1.99	0.82
0.6	2; 2.01; 2.02	0.8
0.74	2.64; 2.65	0.68
0.75	2.66; 2.67; 2.68	0.68
0.76	2.73; 2.74; 2.75	0.66
0.77	2.8; 2.81	0.64
0.78	2.87; 2.88; 2.89	0.64
0.8	3; 3.01	0.6
0.85	3.55	0.53
0.96	7; 7.01; 7.02; 7.03; 7.04; 7.05; 7.06; 7.07	0.28

First of all, solution λ does not exist for each v. Moreover, for some of v solution λ is not unique. And for the found pair $\{v, \lambda\}$ solution x_2 does not always exist. Thus, we can state the following.

Theorem 10.7. *Probability of the existance of Observer's Mathematics Lorentz transformation* *with given* v*,* $0 < v < c$*, in* W_n *is less than* 1.

Let's consider now classical Lorentz transformation effects such as time delay, relativity of simultaneity, and length contraction from point of view Observer's Mathematics Lorentz transformation. Let's start from length contraction. Let's take $n = 2$, $x'_s = 0$, $x'_f = 1$, $t_s = t_f = 0$, $c = 1$, $v = 0.57$, and $\lambda = \mu = 1.95$, where the index s denotes segment origin, and index f means segment end. Put this data into the system 16 to get the following system.

$$\begin{cases} 0 -_2 t'_s = \lambda \times_2 (x_s -_2 0) \\ \mu \times_2 (0 +_2 t'_s) = x_s +_2 0 \end{cases} \tag{22}$$

and $x_s = 0$

$$\begin{cases} 1 -_2 t'_f = \lambda \times_2 (x_f -_2 0) \\ \mu \times_2 (1 +_2 t'_f) = x_f +_2 0 \end{cases} \tag{23}$$

and $x_f = 0.82$. So, we have $x_f -_2 x_s = 0.82 < x'_f -_2 x'_s = 1$, i.e., in this case we have length contraction. Let's now take $n = 2$, $x'_s = 0$, $x'_f = 0.01$, $t_s = t_f = 0$, $c = 1$, $v = 0.57$, and $\lambda = \mu = 1.95$. We get again $x_s = 0$ and the following system.

$$\begin{cases} 0.01 -_2 t'_f = \lambda \times_2 (x_f -_2 0) \\ \mu \times_2 (0.01 +_2 t'_f) = x_f +_2 0 \end{cases} \tag{24}$$

and $x_f = 0.01$. So, we have $x_f -_2 x_s = 0.01 = x'_f -_2 x'_s$, i.e., in this case, there is not length contraction. And finally let's take $n = 2$, $x'_s = 0$, $x'_f = 2.14$, $t_s = t_f = 0$, $c = 1$, $v = 0.57$, and $\lambda = \mu = 1.95$. We get again $x_s = 0$ and the following system.

$$\begin{cases} 2.14 -_2 t'_f = \lambda \times_2 (x_f -_2 0) \\ \mu \times_2 (2.14 +_2 t'_f) = x_f +_2 0 \end{cases} \tag{25}$$

and x_f does not exist. So, we can state

THEOREM 10.8. *In Observer's Mathematics Lorentz transformation, the length of segment* $[x_s, x_f]$ *in coordinate system* K *may:*

- *be less than its length in coordinate system* K',
- *be equal to its length in coordinate system* K',
- *not exist.*

Let's now consider the relativity of simultaneity effect. Let's take $n = 2$, $x_a = 0$, $x_b = 1$, $t_a = t_b = 0$, $c = 1$, $v = 0.57$, and $\lambda = \mu = 1.95$. Put this data into the system 16 to get the following system.

$$\begin{cases} x'_a -_2 t'_a = 0 \\ \mu \times_2 (x'_a +_2 t'_a) = 0 \end{cases} \tag{26}$$

and $t'_a = 0$.

$$\begin{cases} x'_b -_2 t'_b = \lambda \times_2 (x_b) \\ \mu \times_2 (x'_b +_2 t'_b) = x_b \end{cases} \tag{27}$$

and $t'_b = -0.7$. So, $t'_b -_2 t'_a = -0.7 \neq 0$, i.e., in this case, we have the relativity of simultaneity effect. Let's now take $n = 2$, $x_a = 0$, $x_b = 0.01$, $t_a = t_b = 0$, $c = 1$, $v = 0.57$, and $\lambda = \mu = 1.95$. We have in this case $t'_a = 0$ and $t'_b = 0$, i.e., in this case, we do not have the relativity of simultaneity effect. And finally lets take $n = 2$, $x_a = 0$, $x_b = 0.48$, $t_a = t_b = 0$, $c = 1$, $v = 0.57$, and $\lambda = \mu = 1.95$. We get again $t'_a = 0$ and t'_b does not exist. So, we can state the following.

THEOREM 10.9. *In Observer's Mathematics Lorentz transformation simultaneous events in coordinate system* K *may:*

- *not be simultaneous in coordinate system* K',

- *be simultaneous in coordinate system K',*
- *not exist.*

Let's now consider the time delay effect. Let's take $n = 2$, $x'_a = x'_b = 0$, $t_a = 0$, $t_b = 1$, $c = 1$, $v = 0.57$, and $\lambda = \mu = 1.95$. We can calculate: $t'_a = 0$, $t'_b = 0.82$ and $t'_b -_2 t'_a = 0.82 < t_b -_2 t_a = 1$, i.e., in this case, there is time delay. Let's now take $n = 2$, $x'_a = x'_b = 0$, $t_a = 0$, $t_b = 0.01$, $c = 1$, $v = 0.57$, and $\lambda = \mu = 1.95$. We have in this case $t'_a = 0$, $t'_b = 0.01$ and $t'_b -_2 t'_a = 0.01 = t_b -_2 t_a$, i.e., in this case, there is no time delay. And finally let's take $n = 2$, $x'_a = x'_b = 0$, $t_a = 0$, $t_b = 0.48$, $c = 1$, $v = 0.57$, and $\lambda = \mu = 1.95$. We have $t'_a = 0$, t'_b does not exist. So, we can state the following.

THEOREM 10.10. *In Observer's Mathematics Lorentz transformation interval of time on clocks in coordinate system K' may:*

- *be less than interval of time on clocks in coordinate system K,*
- *equal to interval of time on clocks in coordinate system K,*
- *not exist.*

REFERENCES

[1] Khots, B. and Khots, D., [*Mathematics of Relativity*], Web Book (2004).

[2] Khots, B. and Khots, D., "An introduction to mathematics of relativity," in [*Lecture Notes in Theoretical and Mathematical Physics*], Aminova, A., ed., **7**, 269–306 (2006).

[3] Khots, B. and Khots, D., "Classical and quantum mechanics aspects from observer's mathematics point of view," in [*Proceedings of ICM2014*], (2014).

[4] Khots, B. and Khots, D., "Lagrangian in classical mechanics and in special relativity from observer's mathematics point of view," in [*Foundation of Physics, Springer*], **1572-9516** (2015).

[5] Khots, B. and Khots, D., "Special relativity from observer's mathematics point of view," in [*SPIE Proceedings*], **9570**, 9570–49 (2015).

[6] Khots, D., "Euclidean and lobachevsky geometries in mathematics of relativity," in [*Recent problems in Field theory*], Aminova, A., ed., **5**, 243–246 (2006).

[7] Khots, D. and Khots, B., "Non-euclidean geometry in observer's mathematics," in [*Acta Physica Debrecina*], **XLII**, 112–119 (2008).

[8] Khots, D. and Khots, B., "Data mining in observer's mathematics," in [*International Journal of Pure and Applied Mathematics*], **51**, 195–201 (2009).

[9] Khots, B. and Khots, D., "Chaos problems in observer's mathematics," in [*Chaos theory: modeling, simulations and applications, World Scientific*], 215–222 (2011).

[10] Khots, B. and Khots, D., "Observer's mathematics - mathematics of relativity," in [*Applied Mathematics and Computations, Elsevier*], **187**, 228–238 (2007).

[11] Cauchy, A., [*Comptes rend*] (1842).

[12] Kowalevski, S., [*Zur theorie partiellen differentiagleichung*] (1875).

[13] Khots, B. and Khots, D., "Analogy of fermat's last problem in observer's mathematics mathematics of relativity," in [*Proceedings of ICM2006*], (2006).

[14] Khots, B. and Khots, D., "Analogy of hilbert's ten problem in observer's mathematics," in [*Proceedings of ICM2010*], (2010).

[15] Khots, D. and Khots, B., "Fermat's, mersenne's and waring's problems in observer's mathematics," in [*International Journal of Pure and Applied Mathematics*], **43**, 403–408 (2008).

[16] Khots, B. and Khots, D., "Tenth hilbert problem in observer's mathematics," in [*Finite or Infinite Dimensional Complex Analysis andApplications*], 81–85 (2008).

[17] Mossinghoff, M., [*Small Salem numbers, Tabulation*], Web Book, www.math.ucla.edu/mjm/lc/lista/SlemList.html (1996).

[18] Khots, B. and Khots, D., "Probability in quantum theory from observer's mathematics point of view," in [*American Institute of Physics (AIP)*], **1424**, 154–159 (2012).

[19] Khots, B. and Khots, D., "The foundations of quantum mechanics in observer's mathematics," in [*American Institute of Physics (AIP)*], **1508**, 407–411 (2012).

[20] Khots, B. and Khots, D., "Geometrical and quantum mechanical aspects in observer's mathematics," in [*Proceedings of the SPIE Optics and Photonics 2013 Conference 8832 - The Nature of Light - What are Photons? V*], **8832**, 8832 ID–1–8832 ID–6 (2013).

[21] Landau, L. and Lifshitz, E., [*Mechanics, Course of Theoretical Physics, volume 1*] (1969).

[22] Landau, L. and Lifshitz, E., [*The Classical Theory of Fields, Course of Theoretical Physics, volume 2,*] (1972).

[23] Khots, B. and Khots, D., "Quantum mechanics problems in observer's mathematics," in [*American Institute of Physics (AIP)*], **1493**, 518–522 (2012).

[24] Lamb, W. and Scully, M., [*In polarization, Matiere et Rayonnement*] (1969).

[25] Hanbury-Brown, R. and Twiss, R., [*Nature 177, 27-29*] (1956).

[26] Greenstein, G. and Zajone, A., [*The Quantum Challenge*] (2006).

[27] Khots, B. and Khots, D., "Photoelectric effect from observer's mathematics point of view," in [*American Institute of Physics*], **1637**, 487 (2014).

[28] Rashevskii, P., [*Riemannian Geometry and Tenzor Analysis*] (1955).

[29] Khots, B. and Khots, D., "Observer's mathematics applications to the quantum mechanics," in [*The Royal Swedish Academy of Sciences, Physica Scripta*], **2014 T163** (2014).

[30] Khots, B. and Khots, D., "Solitary waves and dispersive equations from observer's mathematics point of view," in [*Geometry in large, topology and aplications*], (2010).

[31] Khots, B. and Khots, D., "What is behind small deviations of quantum mechanics theory from experiments? observer's mathematics point of view," in [*American Institute of Physics*], **1637**, 491 (2014).

[32] Khots, B. and Khots, D., "Observer's mathematics applications to number theory, geometry, analysis, classical and quantum mechanics," in [*Scientific notes of Kazan State University, Physics Mathematics section*], Aminova, A., ed., **153**, 196–203 (2011).

[33] Khots, B. and Khots, D., "Two-slit interference and wave-particle duality for single photons from observer's mathematics point of view," in [*SPIE Proceedings*], **8182**, 8182 H–1–8182 H–6 (2011).

[34] Khots, B. and Khots, D., "Chaos from observer's mathematics point of view," in [*Chaotic systems. Theory amd Applications*], 111–118 (2010).

[35] Einstein, A., [*Relativity: The Special and the General Theory*], Routledge (1920).